

A p -ADIC EISENSTEIN MEASURE FOR UNITARY GROUPS

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ABSTRACT. We construct a p -adic Eisenstein measure with values in the space of p -adic automorphic forms on certain unitary groups. Using this measure, we p -adically interpolate certain special values of both holomorphic and non-holomorphic Eisenstein series, as both the archimedean and the p -adic weights of the Eisenstein series vary.

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1. INTRODUCTION

1.1. Motivating goals.

1.1.1. *Number theory.* This paper completes one step toward a construction of certain p -adic L -functions on unitary groups.

Starting with the work of Jean Pierre Serre and Nicholas Katz, p -adic families of Eisenstein series have been used to construct certain p -adic L -functions. In [Kat78], Katz constructs a p -adic Eisenstein measure for Hilbert modular forms, which he uses to construct p -adic L -functions for CM fields.

Continuing in this direction, we construct a p -adic Eisenstein measure with values in the space of p -adic automorphic forms on particular unitary groups. Like in [Kat78], we p -adically interpolate certain special values of both holomorphic and non-holomorphic Eisenstein series (similar to the Eisenstein series studied by Shimura in [Shi97]), and we allow both the p -adic and the archimedean weights to vary. Unlike in [Kat78] but following the theme of [HLS06], we work with Eisenstein series on adelic groups. The reasons working adelically are:

- (1) An ongoing joint project of Michael Harris, Jian-Shu Li, Christopher Skinner and myself to construct p -adic L -functions uses the doubling method, which requires integrating Eisenstein series on adelic groups [EHLS].
- (2) p -adic interpolation of special values of the Eisenstein series on adelic unitary groups studied extensively by Goro Shimura (e.g. in [Shi97]) constitutes a necessary step in the approach of [HLS06, EHLS].

In [Eis], we generalize the measure in this paper to the case of Eisenstein series with non-scalar weights. However, the results in the current paper are sufficient for allowing the variable conventionally denoted “ s ” in the L -function to vary p -adically. They also are sufficient for the conjectured connection with homotopy theory, described below.

1.1.2. Homotopy theory. Katz’s p -adic Eisenstein series for modular forms are one of the main tools in Matthew Ando, Michael Hopkins, and Charles Rezk’s study of the Witten genus, a modular form-valued invariant of a particular class of manifolds [Hop95, Hop02, AHR10]. This invariant gives an orientation in the cohomology theory *topological modular forms*. (The Witten genus was initially defined by Edward Witten in terms of a certain formal power series and was later studied in terms of p -adic Eisenstein series by Hopkins.)

Mark Behrens, Hopkins, and Niko Naumann aim to generalize the Witten genus to *topological automorphic forms* [Beh09]. They expect p -adic Eisenstein series on unitary groups, and in particular p -adic interpolation of the weight at ∞ , to be a key ingredient in their work. The measure constructed here gives the first example of an Eisenstein measure on unitary groups in which the weight at ∞ varies p -adically. Although the Eisenstein series in this paper are on $U(n, n)$, they can be pulled back in a natural way to $U(n-1, 1)$, which is the group currently of interest in homotopy theory.

1.2. Main results. The main result of this paper is the construction in Section 4 of a p -adic measure with values in the space of p -adic

automorphic forms on the unitary groups $U(n, n)$. As part of the construction of this measure, we construct a p -adic family of Eisenstein series. As a corollary, we obtain a p -adic measure that p -adically interpolates certain special values of Eisenstein series (normalized by an appropriate period) initially defined over \mathbb{C} , including non-holomorphic Eisenstein series.

In Sections 2 and 3, we introduce the Eisenstein series with which we work. In the C^∞ -case, these are similar to the Eisenstein series on adelic groups studied by Shimura in [Shi97]. The main change to Shimura's approach, like in [HLS06], is at primes dividing p . As explained below, it was necessary to modify the construction at p in [HLS06]. Note that Lemma 10 of the current paper provides a formulaic approach to constructing a whole class of Siegel sections at p that are well-suited to p -adic interpolation of the Fourier coefficients of the Eisenstein series.

Our approach to p -adic Eisenstein series and construction of the p -adic Eisenstein measure is a natural generalization of Katz's methods in [Kat78]. Although the formulas in our paper look more complicated than the formulas derived by Katz in [Kat78], the reader can plug in $n = 1$ to verify easily that we really are studying similar Eisenstein series to those in [Kat78] in that case. (Indeed, a fair amount of cancellation occurs in the terms in the Fourier coefficients precisely when $n = 1$, and the indexing is also simpler in that case.)

1.3. Relationship with prior results. The current paper extends, clarifies, and corrects portions of the construction of the measure in [HLS06]. Unlike the construction of [HLS06], the measure in the current paper allows

- (1) p -adic interpolation of special values of both holomorphic and non-holomorphic Eisenstein series, as the weight of the Eisenstein series varies p -adically.
- (2) p -adically varying the weight of the Eisenstein series at ∞ in addition to p .

These were not goals of the construction in [HLS06]. These generalizations rely on the differential operators in [Eis12] and a close adaptation of the approach to Hecke characters in [Kat78, Section 5].

1.3.1. Clarifications and corrections to [HLS06]. During the course of this work, I came across several issues in the work in [HLS06]. At Skinner's recommendation, I take this opportunity to give corrections and clarifications to the following points.

- (1) This paper provides a few corrections to the Fourier coefficients in [HLS06], which are computed as a product of local Fourier

coefficients in both [HLS06] and the current paper. In the notation of [HLS06], the most important changes are:

- (a) At archimedean primes, the term $\det \beta^{[E:\mathbb{Q}]}$ in the local Fourier coefficient in [HLS06, Equation (3.3.5.3)] should be $N_{E/\mathbb{Q}}(\beta)$. (Shimura's computation of the local Fourier coefficients shows that each of those terms is of the form $\sigma(\det \beta)$, where σ varies over all the embeddings of the totally real field E into $\bar{\mathbb{Q}}$ [Shi97, Shi82].) We address this change in Warning 8 and Section 2.2.6.
- (b) At primes v dividing p , the Siegel section in [HLS06, Equation (3.3.4.7)] appears not to be a section of the induced representation $\mathbf{Ind}_{P_v}^{G_v}(\chi|\cdot|^{-s})$, based on evaluation of this section at elements of P_v . (It is possible there is a minor modification that would make it into a section of the induced representation.)

Additionally, the differential operators in the current paper and the p -adic L -functions in [EHLS] require a p -adic section with further properties than those necessary for the Eisenstein measure in [HLS06]. Section 2.2.8 of the current paper provides a class of p -adic sections that satisfies these further properties.

- (c) Away from p and ∞ , some of the Fourier coefficients are in terms of polynomials that actually vary as the Hecke character varies. However, [HLS06] seems to treat the values of these polynomials as invariant under p -adic variation of the Hecke character. (See Section 2.2.9 of the current paper for formulas for the Fourier coefficients at these places, similar to those given in [HLS06].) This is closely related to the next item, which concerns variation of the Hecke character.
- (2) Let χ be a Hecke character. The approach in [HLS06, Section (3.5)] seems to be to fix the archimedean component of a Hecke character on a CM field K and vary just the p -adic component. However, changing only the p -adic component of a Hecke character might give a character of the idèle group \mathbb{A}_K^\times that is no longer trivial on K^\times . Instead, we use a similar approach to [Kat78], which is also necessary for varying the archimedean component of the Eisenstein series in this paper.

Our method, which is based on the approach in [Kat78, Sections 5.5, 5.0], seems only to allow for a construction in which both the p -adic and the archimedean components of the weights

of the Eisenstein series can vary p -adically. See Sections 2.2.13 and 4 for further details on our approach.

- (3) As a consequence of (2), the measure in [HLS06, Section (3.5)] should have been constructed on the product of the group denoted $T(l)$ with the *quotient* $(\mathcal{O}_K \otimes \mathbb{Z}_p) / \overline{\mathcal{O}_K^\times}$, where $\overline{\mathcal{O}_K^\times}$ denotes the p -adic closure of \mathcal{O}_K^\times .
- (4) As in the convention for modular forms, our q -expansions have coefficients that agree over \mathbb{C} with the Fourier coefficients of a function of a complex variable in a hermitian symmetric space (e.g. the upper half plane and its generalizations). This requires normalizing the Eisenstein series on adelic groups by an automorphy factor. This normalization is missing in [HLS06], which accounts for some of the difference between the q -expansion coefficients used in [HLS06, Theorem (3.5.1)] and in Equation (29) here. (The normalization in this paper is necessary for applying the q -expansion principle to the q -expansions, which is necessary for p -adically interpolating the Eisenstein series.)

1.4. Acknowledgments. I am grateful to Matthew Emerton and Christopher Skinner for insightful suggestions during the past year, especially as I came across modifications that I needed to make to [HLS06]. Many of the suggestions found their way into Section 2.2.8. I am grateful to Michael Harris for helpful discussions, responses to my questions, and encouragement to complete this project, as well as enthusiastic suggestions for subsequent related projects.

I thank Mark Behrens for answering my questions about the conjectured implications of the p -adic Eisenstein series for homotopy theory. I would also like to thank both Harris and Behrens for alerting me, in the first place, to the role of p -adic Eisenstein series in homotopy theory.

This project relies upon the ideas in Nicholas Katz's construction of Eisenstein measures, without which the current project would be impossible. This paper also relies heavily upon the helpfully detailed presentation of C^∞ -Eisenstein series in Goro Shimura's papers; if Shimura's writing had not been so precise, I would have struggled much more with the non- p -adic portion of this project.

2. SIEGEL EISENSTEIN SERIES ON CERTAIN UNITARY GROUPS

2.1. Unitary groups. We now introduce the unitary groups with which we work throughout this paper. The material in this section is a special case of the material in [Shi97, Section 21], and the setup is also similar to that in [Eis12] and [HLS06].

Fix a CM field K , and denote its ring of integers by \mathcal{O}_K . Let V be a vector space of dimension n over K , and let $\langle v_1, v_2 \rangle_V$ be a non-degenerate hermitian pairing on V . Let $-V$ denote the vector space V with the hermitian pairing $-\langle v_1, v_2 \rangle_V$, and let

$$\begin{aligned} W &= 2V = V \oplus -V \\ \langle (v_1, v_2), (v'_1, v'_2) \rangle_W &= \langle v_1, v'_1 \rangle_V + \langle v_2, v'_2 \rangle_{-V} \\ &= \langle v_1, v'_1 \rangle_V - \langle v_2, v'_2 \rangle_V. \end{aligned}$$

Then the unitary group $G = U(W) = \{g | \langle gv, gw \rangle = \langle v, w \rangle \forall v, w \in W\}$ is of signature (n, n) . Note that the canonical embedding

$$V \oplus V \hookrightarrow W$$

induces an embedding

$$U(V) \times U(-V) \hookrightarrow U(W).$$

We write $W = V_d \oplus V^d$, where V_d and V^d denote the maximal isotropic subspaces

$$\begin{aligned} V^d &= \{(v, v) | v \in V\} \\ V_d &= \{(v, -v) | v \in V\}. \end{aligned}$$

Let P be the Siegel parabolic subgroup of G stabilizing V^d in $V_d \oplus V^d$ under the action of G on the right. Denote by M the Levi subgroup of P and by N the unipotent radical of P .

Two convenient choices of bases. Let e_1, \dots, e_n be an orthogonal basis for V and ϕ be the matrix for $\langle \cdot, \cdot \rangle_V$ with respect to e_1, \dots, e_n . For $i = 1, \dots, n$, let $e_{i+n} = e_i$. Then the matrix for $\langle \cdot, \cdot \rangle_{2V}$ with respect to the basis e_1, \dots, e_{2n} is

$$\omega = \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}.$$

With respect to the basis e_1, \dots, e_{2n} , $(g, h) \in U(V) \times U(-V)$ is the element $\text{diag}(g, h) \in U(2V)$. Let α be a totally imaginary element of K prime to p , and let

$$(1) \quad \begin{aligned} \eta &= \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \\ S &= \begin{pmatrix} 1_n & -\frac{\alpha}{2}\phi \\ -1_n & -\frac{\alpha}{2}\phi \end{pmatrix}. \end{aligned}$$

Note that $S\eta^t\bar{S} = \alpha\omega$. Using the base change given by S , we see that $U(2V)$ is isomorphic to

$$G(\eta) = \{g | g\eta^t\bar{g} = \eta\}.$$

In particular,

$$S^{-1}G(\omega)S = G(\eta),$$

and

$$\begin{aligned} (g, h) \mapsto S^{-1}\text{diag}(g, h)S &= \begin{pmatrix} \frac{g+h}{2} & \frac{-\alpha(g-h)\phi}{\frac{4}{\phi^{-1}(g+h)\phi}} \\ -\alpha^{-1}\phi^{-1}(g-h) & \frac{\phi^{-1}(g+h)\phi}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{g+h}{2} & \frac{-\alpha(g-h)\phi}{\frac{4}{\phi^{-1}(g+h)\phi}} \\ -\alpha^{-1}\phi^{-1}(g-h) & \frac{\phi^{-1}(g+h)\phi}{2} \end{pmatrix} \in G(\eta) \end{aligned}$$

gives an embedding of $U(V) \times U(-V)$ into $G(\eta)$. Furthermore, $S^{-1}P(\omega)S = P$, where $P = P(\eta)$ is the parabolic subgroup of $G(\eta)$ stabilizing V^d in $V_d \oplus V^d$ under the action of G on the right, and $P(\omega)$ is the stabilizer in $G(\omega)$ of $V^d = \{(v, v) | v \in V\}$ inside of $V \oplus -V$ under action on the right. The Eisenstein series in this paper are on $G(\eta)$ with respect to the parabolic subgroup $P = P(\eta)$ stabilizing V^d (so they are Siegel Eisenstein series).

Note that the choice of basis for V over K fixes an isomorphism

$$(2) \quad M \xrightarrow{\sim} \text{GL}_K(V),$$

identifying M with subgroup of G consisting of all elements of the form $\text{diag}({}^t\bar{h}^{-1}, h)$ with h in $\text{GL}_K(V)$.

Choice of a Shimura datum. Let E be the maximal totally real subextension of the CM field K . Our setup is as in [HLS06, Section 1.2]; we review the details relevant to our construction. We fix a Shimura datum $(GU(2V), X(2V))$, and a corresponding Shimura variety $Sh(W)$, according to the conditions in [HLS06] and in [Eis12]. Similarly, we attach Shimura data and Shimura varieties to $G(U(V) \times U(V))$. Note that the symmetric domain $X(2V)$ is holomorphically isomorphic to the tube domain \mathcal{H}_n consisting of $[E : \mathbb{Q}]$ copies of

$$\{z \in M_{n \times n}(\mathbb{C}) | i({}^t\bar{z} - z) > 0\}.$$

Let \mathcal{K}_∞ be the stabilizer in $G(\mathbb{R})$ of a fixed point

$$(3) \quad \mathfrak{1} = (\mathfrak{1}_\sigma)$$

in $\prod_{\sigma: E \hookrightarrow \bar{\mathbb{Q}}} \mathcal{H}_n$, so we can identify $G(\mathbb{R})/\mathcal{K}_\infty$ with \mathcal{H}_n . Given a compact open subgroup \mathcal{K} of $G(\mathbb{A}_f)$, denote by ${}_\mathcal{K}Sh(W)$ the Shimura variety whose complex points are given by

$$G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / \mathcal{K}.$$

This Shimura variety is a moduli space for abelian varieties together with a polarization, an endomorphism, and a level structure (dependent upon the choice of \mathcal{K}).

Relationship between automorphic forms on \mathcal{H}_n and on $G(\eta)$. Automorphic forms $f(g)$ of weight ρ viewed as functions on $G(\eta)$ and automorphic forms $\tilde{f}(z)$ of weight ρ viewed as functions on \mathcal{H}_n are related via

$$f(g) \leftrightarrow \rho((C\mathfrak{I} + D), (\bar{C}^t\mathfrak{I} + \bar{D}))\tilde{f}(g\mathfrak{I}),$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(\mathbb{R})$.

As explained precisely in [Shi97, Section 22], an automorphic form f on $G(\eta)$ pulls back to an automorphic form $f(g, h)$ on $U(V) \times U(-V)$ which satisfies an automorphy property in terms of g (independent of h) and an automorphy property in terms of h (independent of g), which at each factor is completely determined by the automorphy property of f on $G(\eta)$. (The details of the spaces on which $G(V)$ acts are unnecessary for the current paper. They are discussed in [Shi97, Sections 22, 6].) In particular, a p -adic family of Eisenstein series on $G(\eta)$ pulls back to a p -adic family of automorphic forms on $U(V) \times U(-V)$ (parametrized by the weight of the Eisenstein series).

2.2. Certain Eisenstein series. Choice of a CM type. Let p be a rational prime that is unramified in K and such that each prime of the maximal totally real subfield $E \subseteq K$ dividing p splits completely in the CM field K . Fix embeddings

$$\begin{aligned} \iota_\infty : \bar{\mathbb{Q}} &\hookrightarrow \mathbb{C} \\ \iota_p : \bar{\mathbb{Q}} &\hookrightarrow \bar{\mathbb{Q}}_p, \end{aligned}$$

and fix an isomorphism

$$\iota : \bar{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$$

satisfying $\iota \circ \iota_p = \iota_\infty$. From here on, we identify $\bar{\mathbb{Q}}$ with $\iota_p(\bar{\mathbb{Q}})$ and $\iota_\infty(\bar{\mathbb{Q}})$.

Fix a CM type Σ for K/\mathbb{Q} . For each element $\sigma \in \text{Hom}(E, \bar{\mathbb{Q}})$, we also write σ to denote the unique element of Σ prolonging $\sigma : E \hookrightarrow \bar{\mathbb{Q}}$ (when no confusion can arise). For each element $x \in K$, denote by \bar{x} the image of x under the unique non-trivial element $\epsilon \in \text{Gal}(K/E)$, and let $\bar{\sigma} = \sigma \circ \epsilon$.

Given an element a of F , we identify it with an element of $F \otimes \mathbb{R}$ via the embedding

$$\begin{aligned} (4) \quad F &\hookrightarrow F \otimes \mathbb{R} \\ (5) \quad a &\mapsto (\sigma(a))_{\sigma \in \Sigma}. \end{aligned}$$

We identify $a \in K$ with an element of $K \otimes \mathbb{C} \xrightarrow{\sim} (F \otimes \mathbb{C}) \times (F \otimes \mathbb{C})$ via the embedding

$$\begin{aligned} K &\hookrightarrow K \otimes \mathbb{C} \\ a &\mapsto ((\sigma(a))_{\sigma \in \Sigma}, (\bar{\sigma}(a))_{\sigma \in \Sigma}). \end{aligned}$$

2.2.1. *A unitary Hecke character of type A_0 .* Let \mathfrak{m} be an integral ideal in the totally real field \mathcal{O}_E that divides p^∞ . Let χ be a unitary Hecke character of type A_0

$$\chi : \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$$

of conductor \mathfrak{m} , i.e.

$$\chi_v(a) = 1$$

for all finite primes v in K and all $a \in K_v^\times$ such that

$$a \in 1 + \mathfrak{m}_v \mathcal{O}_{K_v}.$$

Let $\nu(\sigma)$ and $k(\sigma)$, $\sigma \in \Sigma$, denote integers such that the infinity type of χ is given by

$$(6) \quad \prod_{\sigma \in \Sigma} \sigma(b)^{-k(\sigma)-2\nu(\sigma)} (\sigma(b)\bar{\sigma}(b))^{\frac{k(\sigma)}{2}+\nu(\sigma)},$$

i.e. χ is of infinity-type

$$0 \cdot \sum_{\sigma \in \Sigma} \sigma - \sum_{\sigma \in \Sigma} \left(\frac{k(\sigma)}{2} + \nu(\sigma) \right) (\sigma - \bar{\sigma}).$$

Remark 1. Given an element $a \in K$, we associate a with an element of $K \otimes \mathbb{R}$, via the embedding

$$a \mapsto (\sigma(a))_{\sigma \in \Sigma},$$

so for $a \in K$,

$$\left(\prod_{v|\infty} \chi_v \right) (a) = \prod_{\sigma \in \Sigma} \chi_v(\sigma_v(a)).$$

Conventions for adelic norms. Let $|\cdot|_E$ denote the adelic norm on $E^\times \backslash \mathbb{A}_E^\times$ such that for all $a \in \mathbb{A}_E^\times$,

$$|a|_E = \prod_v |a|_v,$$

where the product is over all places of E and where the absolute values are normalized so that

$$\begin{aligned} |v|_v &= q^{-1}, \\ q &= \text{the cardinality of } \mathcal{O}_{E_v}/v\mathcal{O}_{E_v}, \end{aligned}$$

for all non-archimedean primes v of the totally real field E . Consequently for all $a \in E$,

$$\prod_{v|\infty} |a|_v = \prod_{v \in \Sigma} \sigma_v(a),$$

where the product is over all archimedean places v of the totally real field E . We denote by $|\cdot|_K$ the adelic norm on $K^\times \backslash \mathbb{A}_K^\times$ such that for all $a \in \mathbb{A}_K^\times$,

$$|a|_K = |a\bar{a}|_E.$$

For $a \in K$ and v a place of E , we let

$$|a|_v = |a\bar{a}|_v^{\frac{1}{2}}.$$

2.2.2. Siegel Eisenstein series. Let χ be a unitary Hecke character that meets the conditions of Section 2.2.1. For any $s \in \mathbb{C}$, we view $\chi \cdot |\cdot|_K^{-s}$ as a character of the parabolic subgroup $P(\mathbb{A}_E) = M(\mathbb{A}_E)N(\mathbb{A}_E)$ via the composition of maps

$$P \xrightarrow{\text{mod } N} M \xrightarrow{\text{the iso in (2)}} \text{GL}_K(V) \xrightarrow{\text{iso fixed in Section 2.1}} \text{GL}_n(K) \xrightarrow{\det} \mathbb{C}^\times \xrightarrow{\chi \cdot |\cdot|_K^{-s}} \mathbb{C}^\times.$$

Consider the induced representation

$$(7) \quad I(\chi, s) = \mathbf{Ind}_{P(\mathbb{A}_E)}^{G(\mathbb{A}_E)} (\chi \cdot |\cdot|_K^{-s}) \xrightarrow{\sim} \otimes_v \mathbf{Ind}_{P(E_v)}^{G(E_v)} (\chi_v |\cdot|_v^{-2s}),$$

where the product is over all places of E .

Remark 2. Although it is in some contexts conventional to use normalized induction and consider $\chi \cdot |\cdot|_K^{s-n/2}$ instead of $\chi \cdot |\cdot|_K^s$, we leave out the factor $|\cdot|_K^{-n/2}$ for now. (It just comes from the modulus character and will reappear when we compute the Fourier coefficients.) The reader who prefers to include $n/2$ in this expression may just regard it as absorbed into s .

Given a section $f \in I(\chi, s)$, the Siegel Eisenstein series associated to f is the \mathbb{C} -valued function of G defined by

$$E_f(g) = \sum_{\gamma \in P(E) \backslash G(E)} f(\gamma g)$$

This function converges for $\Re(s) > 0$ and can be continued meromorphically to the entire plane.

Remark 3. If we were working with normalized induction, then the function would converge for $\Re(s) > \frac{n}{2}$, but we have absorbed the exponent $\frac{n}{2}$ into the exponent s . (Our choice not to include the modulus character at this point is equivalent to shifting the plane on which the function converges by $\frac{n}{2}$.)

All the poles of E_f are simple and there are at most finitely many of them. Details about the poles are given in [Tan99].

2.2.3. Conditions imposed on the choice of a Siegel section. We continue to work with a unitary Hecke character χ meeting the conditions of Section 2.2.1, which has infinity type given by (6). Let $k(\sigma)$ and $\nu(\sigma)$ be as in (6).

In order to construct an Eisenstein series that behaves nicely with respect to p -adic interpolation, we shall work with a section $f \in I(\chi, s)$ that meets the following conditions (among other conditions that we will specify as they become relevant):

Condition 4. There is a compact open subgroup U of $G(\mathbb{A}_E)$ and a \mathbb{Q}^\times -valued character ρ of $\prod_{v|p} U_v$ such that for all $u \in U$ and $h \in G(\mathbb{A}_E)$,

$$E_f(hu) = \rho((u_v)_{v|p}) J_{(u_v)_{v|p}}^{k, \nu}(\mathbf{1})^{-1} E_f(h),$$

where $k = (k(\sigma))_{\sigma \in \Sigma}$, $\nu = (\nu(\sigma))_{\sigma \in \Sigma}$, and $J_\alpha^{k, \nu}(z)$ is defined for all $z \in \mathcal{H}_n$ and

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \right)_{\sigma \in \Sigma} \in \prod_{\sigma| \infty} G(\mathbb{R})$$

by

$$(8) \quad J_\alpha^{k, \nu}(z) := \prod_{\sigma \in \Sigma} (\det \alpha_\sigma)^{\nu(\sigma)} \det(c_\sigma z_\sigma + d_\sigma)^{k(\sigma)} \\ = \prod_{\sigma \in \Sigma} \det(c_\sigma z_\sigma + d_\sigma)^{k(\sigma) + \nu(\sigma)} \prod_{\sigma \in \Sigma} \det(\bar{c}_\sigma^t z_\sigma + \bar{d}_\sigma)^{-\nu(\sigma)}.$$

Condition 5. The section f factors as $f = \otimes_v f_v$ with $f_v \in \mathbf{Ind}_{P(E_v)}^{G(E_v)}(\chi_v | \cdot |_v^{2s})$ for each place v of E . (This condition allows us to express the Fourier coefficients as a product of local factors, which is important for our approach to p -adic interpolation of the the Fourier coefficients.)

Remark 6. Many different choices for the local sections f_v lend themselves nicely to construction of a p -adic Eisenstein measure. The most constrained choices are at p and at ∞ .

2.2.4. Preliminaries on Fourier expansions. Exponential characters. Before explaining the consequences of Conditions 4 and 5, we need to introduce some conventions for exponential characters. For each archimedean place $v \in \Sigma$, denote by \mathbf{e}_v the character of E_v (i.e \mathbb{R}) defined by

$$\mathbf{e}_v(x_v) = e^{(2\pi i x_v)}$$

for all x_v in E_v . Denote by \mathbf{e}_∞ the character of $E \otimes \mathbb{R}$ defined by

$$\mathbf{e}_\infty((x_v)_{v \in \Sigma}) = \prod_{v \mid \infty} \mathbf{e}_v(x_v).$$

Following our convention from (4), we put

$$\mathbf{e}_\infty(a) = \mathbf{e}_\infty((\sigma(a))_{\sigma \in \Sigma}) = \mathbf{e}^{2\pi i \operatorname{tr}_{E/\mathbb{Q}}(a)}$$

for all $a \in E$. For each finite place v of E dividing a prime q of \mathbb{Z} , denote by \mathbf{e}_v the character of E_v defined for each $x_v \in E_v$ by

$$\mathbf{e}_v(x_v) = e^{-2\pi i y}$$

where y is an element of \mathbb{Q} such that $\operatorname{tr}_{E_v/\mathbb{Q}_q}(x_v) - y \in \mathbb{Z}_p$. We denote by $\mathbf{e}_{\mathbb{A}_E}$ the character of \mathbb{A}_E defined by

$$\mathbf{e}_{\mathbb{A}_E}(x) = \prod_v \mathbf{e}_v(x_v)$$

for all $x = (x_v) \in \mathbb{A}_E$.

Remark 7. Note that for a $a \in E$, we identify a with the element $(\sigma_v(a))_v \in \mathbb{A}_E$, where $\sigma_v : F \hookrightarrow F_v$ is the embedding corresponding to v . Following this convention, we put

$$(9) \quad \mathbf{e}_{\mathbb{A}_E}(a) = \prod_v \mathbf{e}_v(\sigma_v(a)).$$

for all $a \in F$.

Consequences of Conditions 4 and 5. For any subring R of $K \otimes_F F_v$, with v a place of F , let $\operatorname{Her}_n(R)$ denote the space of $n \times n$ -matrices with entries in R .

By [Shi97, Proposition 18.3], Condition 4 guarantees that E_f has a Fourier expansion such that for all $h \in \operatorname{GL}_n(K)$ and $m \in \operatorname{Her}_n(K)$

$$E_f \left(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t \bar{h}^{-1} & 0 \\ 0 & h \end{pmatrix} \right) = \sum_{\beta \in \operatorname{Her}_n(K)} c(\beta, h; f) \mathbf{e}_{\mathbb{A}_E}(\operatorname{tr}(\beta m)),$$

with $c(\beta, h; f)$ a complex number dependent only on the choice of section f , the hermitian matrix $\beta \in \operatorname{Her}_n(K)$, h_v for finite places v , and $(h \cdot {}^t \bar{h})_v$ for archimedean places v of E .

By [Shi97, Sections 18.9, 18.10] and Condition 5, the Fourier coefficient is a product of local Fourier coefficients determined by the local

sections f_v . More precisely, for each $\beta \in \text{Her}_n(K)$,

$$\begin{aligned}
 c(\beta, h; f) &= C(n, K) \prod_v c_v(\beta_v, h_v; f) \\
 c_v(\beta, h; f) &= \\
 &\prod_v \int_{\text{Her}_n(K \otimes_F F_v)} f_v \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & m_v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t \bar{h}_v^{-1} & 0 \\ 0 & h_v \end{pmatrix} \right) \mathbf{e}_v(-\text{tr}(\beta_v m_v)) dm_v \\
 (10) \quad C(n, K) &= 2^{n(n-1)[E:\mathbb{Q}]/2} |D_F|^{-n/2} |D_K|^{-n(n-1)/4},
 \end{aligned}$$

where D_F and D_K are the discriminants of K and F respectively, $\beta_v = \sigma_v(\beta)$ for each place v of F , and d_v denotes the Haar measure on $\text{Her}_n(K_v)$ such that:

$$\begin{aligned}
 &\int_{\text{Her}_n(\mathcal{O}_K \otimes_F F_v)} d_v x = 1, \text{ for each finite place } v \text{ of } F \\
 (11) \quad d_v x &:= \left| \bigwedge_{j=1}^n dx_{jj} \bigwedge_{j < k} (2^{-1} dx_{jk} \wedge d\bar{x}_{jk}) \right|, \text{ for each archimedean place } v \text{ of } F.
 \end{aligned}$$

(In Equation (11), x denotes the matrix whose ij -th entry is x_{ij} .)

Warning 8. *Given $\beta \in \text{Her}_n(K)$, we shall sometimes write $\mathbf{e}_v(\beta)$ when we mean $\mathbf{e}_v(\sigma_v(\beta))$, in keeping with the convention of Equation (9). (This agrees with Shimura's convention in [Shi97] and [Shi82].) Note that this notational ambiguity in Shimura's papers seems to have led to an error in (3.3.5.3) and in computation of the Fourier coefficient at ∞ in [HLS06]. In particular, the factor $\det \beta^{[E:\mathbb{Q}]}$ in [HLS06, Equation (3.3.5.3)] should be $\det(\prod_{\sigma \in \Sigma} \sigma(\beta))$. This correction is necessary for our construction of the Eisenstein measure.*

Fourier coefficients at points on the Levi subgroup. We continue to work with a section $f = \otimes_v f_v$ that lies in the induced representation (7) and meets all the above conditions.

Lemma 9. *For each $h \in \text{GL}_n(\mathbb{A}_K)$ and $\beta \in \text{Her}_n(K)$,*

$$(12) \quad c(\beta, h; f) = \chi({}^t \bar{h}^{-1}) |\det({}^t \bar{h}^{-1} \cdot h^{-1})|_E^{n-s} c({}^t \bar{h}^{-1} \beta h^{-1}, 1_n; f).$$

Proof. As in Equation (1), let $\eta = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. Let $m(h)$ denote the matrix $\begin{pmatrix} {}^t\bar{h}^{-1} & 0 \\ 0 & h \end{pmatrix}$. Observe that for any $n \times n$ matrix m ,

$$\begin{aligned} \eta \cdot m(h) \cdot \eta^{-1} &= m({}^t\bar{h}^{-1}) \\ m(h)^{-1} \cdot \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot m(h) &= \begin{pmatrix} 1 & {}^t\bar{h}mh \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \eta \cdot \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot m(h) &= (\eta \cdot m(h) \cdot \eta^{-1}) \eta \left(m(h)^{-1} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} m(h) \right) \\ &= m({}^t\bar{h}^{-1}) \eta \begin{pmatrix} 1 & {}^t\bar{h}mh \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

So for any place v of E and section $f_v \in \mathbf{Ind}_{P(E_v)}^{G(E_v)}(\chi, s)$,

$$(13) \quad f_v \left(\eta \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} m(h_v) \right) = \chi_v \left({}^t\bar{h}_v^{-1} \right) \left| {}^t\bar{h}_v^{-1} \right|_v^{-2s} f_v \left(\eta \begin{pmatrix} 1 & {}^t\bar{h}_v m h_v \\ 0 & 1 \end{pmatrix} \right)$$

The lemma now follows from Equation (13) and the fact that the Haar measure d_v satisfies $d_v(h_v x {}^t\bar{h}_v) = |\det({}^t\bar{h}_v \cdot h_v)|_v^n d_v(x)$ for each place v of E . \blacksquare

2.2.5. The sections at archimedean places of E . As in Condition 4, let $k = (k(\sigma))_{\sigma \in \Sigma}$ and $\nu = (\nu(\sigma))_{\sigma \in \Sigma}$, and let $J_\alpha^{k, \nu}$ be defined as in Equation (8). Given a unitary Hecke character χ meeting the conditions of Section 2.2.1, a complex number s , and \mathfrak{l} as in Equation (3), we will always take the section at the archimedean places to be of the form

$$f_\infty^{k, \nu}(\bullet; \mathfrak{l}, \chi, s) \in \otimes_{v|\infty} \mathbf{Ind}_{P(E_v)}^{G(E_v)}(\chi_v ||_v^{-2s}),$$

where

$$\begin{aligned} f_\infty^{k, \nu}(\alpha; \mathfrak{l}, \chi, s) &= J_\alpha^{k, \nu}(\mathfrak{l})^{-1} (J_\alpha(\mathfrak{l}) \overline{J_\alpha(\mathfrak{l})})^{\frac{k}{2}-s}, \\ \left(J_\alpha(z) \overline{J_\alpha(z)} \right)^{\frac{k}{2}-s} &:= \prod_{\sigma \in \Sigma} \left(\det(c_\sigma z_\sigma + d_\sigma) \overline{\det(c_\sigma z_\sigma + d_\sigma)} \right)^{\frac{k(\sigma)}{2}-s}, \end{aligned}$$

for each $\alpha = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \in \prod_{v|\infty} G(E_v)$. The relationship between $f_\infty^{k, \nu}(\alpha; \mathfrak{l}, \chi, s)$

and $f_\infty^{k, \nu}(\alpha; i \cdot 1_n, \chi, s)$ is given by

$$(14) \quad f_\infty^{k, \nu}(\alpha; \mathfrak{l}, \chi, s) = f_\infty^{k, \nu}(\alpha g^{-1}; i 1_n, \chi, s) f_\infty^{k, \nu}(g^{-1}; i 1_n, \chi, s)^{-1},$$

where g is any element of G such that $g\mathfrak{l} = i 1_n$.

2.2.6. *The Fourier coefficients at archimedean places of E .* Recall that by Equation (12), the Fourier coefficients $c(\beta, h; f)$ are completely determined by their values at $h = 1_n$. Following Shimura, we give the Fourier coefficients in terms of the section $f_\infty^{k,\nu}(\bullet; i1_n, \chi, s)$ and note that Equation (14) relates the Fourier coefficients in terms of $f_\infty^{k,\nu}(\bullet; \mathfrak{l}, \chi, s) = \otimes_v f_v^{k,\nu}$ to those in terms of $f_\infty^{k,\nu}(\bullet; i1_n, \chi, s)$. The non-zero Fourier coefficients of the Eisenstein series at the archimedean primes, evaluated at $m(1_n)$ are given by the following product over all archimedean places of E :

$$\begin{aligned} & \prod_{v|\infty} c_v(\beta, 1_n; f_v^{k,\nu}(\bullet; i1_n, \chi, s)) \\ &= \prod_{v|\infty} \int_{\text{Her}_n(\mathbb{C})} f_v^{k,\nu} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & m_v \\ 0 & 1 \end{pmatrix} \right) \mathbf{e}(-\text{tr}(\sigma_v(\beta)m_v)) dm_v \\ &= \prod_{v|\infty} \int_{\text{Her}_n(\mathbb{C})} f_v^{k,\nu} \left(\begin{pmatrix} 0 & -1 \\ 1 & m_v \end{pmatrix} \right) \mathbf{e}(-\text{tr}(\sigma_v(\beta)m_v)) dm_v \\ &= \prod_{v|\infty} \int_{\text{Her}_n(\mathbb{C})} \det(i + m_v)^{-\frac{k(\sigma_v)}{2}-s} \det(-i + m_v)^{\frac{k(\sigma_v)}{2}-s} \mathbf{e}(\text{tr}(-\sigma_v(\beta)m_v)) dm_v. \end{aligned}$$

By [Shi97, Lemma 18.12], which is proved in [Shi82], each of the above local integrals converges at least for $\Re\left(\frac{k(\sigma)}{2} + s\right) > 2n - 1$ and can be meromorphically continued to a meromorphic function of $(k(\sigma), s) \in \mathbb{C} \times \mathbb{C}$. When there is an integer k such that

$$s = \frac{k}{2} = \frac{k(\sigma)}{2} \text{ for all } \sigma \in \Sigma$$

(i.e. when $J_\alpha^{k,\nu}(z)^{-1}(J_\alpha(z)\overline{J_\alpha(z)})^{\frac{k}{2}-s}$ is a holomorphic function of $z \in \mathcal{H}_n$), [Shi83, Equation (7.12)] describes the archimedean Fourier coefficients precisely:

$$\begin{aligned} & c_v \left(\beta, 1_n; f_v^{k,\nu} \left(\bullet; i1_n, \chi, \frac{k}{2} \right) \right) \\ (15) \quad &= 2^{(1-n)n} i^{-nk} (2\pi)^{nk} \left(\pi^{n(n-1)/2} \prod_{t=0}^{n-1} \Gamma(k-t) \right)^{-1} \sigma_v(\det \beta)^{k-n} \mathbf{e}(i \text{tr}(\sigma_v(\beta))), \end{aligned}$$

for each archimedean place v of F . Observe that when $k > n$,

$$\prod_{v|\infty} c_v \left(\beta, h; f_v^{k,\nu} \left(\bullet; i1_n, \chi, \frac{k}{2} \right) \right) = 0,$$

unless $\det(\beta) \neq 0$ and $\det(h) \neq 0$, i.e. unless β is of rank n .

2.2.7. Conventions at places of E dividing p . We now establish certain conventions that we will use in Section 2.2.8, when we define the Siegel sections at places of E dividing p .

CM Types. As explained in [Kat78, Section (5.1.10)], the choice of the CM type Σ is equivalent to the choice of a prime in K over each prime in E dividing p . We write $v \in \Sigma$ to mean v is a prime in K dividing p that induces a p -adic embedding $\iota_p \circ \sigma$ for some embedding $\sigma \in \Sigma$, and we associate $v \in \Sigma$ with this embedding $\sigma \in \Sigma$. When no confusion can arise, we write v to denote the prime in Σ lying above a prime v in E dividing p . For each element $x \in K$, denote by \bar{x} the image of x under the unique non-trivial element of $\epsilon \in \text{Gal}(K/E)$, let $\bar{v} = \epsilon(v)$, and let $\bar{\sigma} = \sigma \circ \epsilon$. Let

$$(16) \quad \phi_\Sigma : K \otimes \mathbb{Q}_p \rightarrow E \otimes \mathbb{Q}_p$$

be the projection obtained through the composition of maps

$$K \otimes \mathbb{Q}_p \rightarrow \prod_{v \in \Sigma} (v\text{-adic completion of } K) \xrightarrow{\sim} \prod_{v \in \Sigma} (v\text{-adic completion of } E) = E \otimes \mathbb{Q}_p$$

From here on, we identify $K \otimes \mathbb{Q}_p$ with $(E \otimes \mathbb{Q}_p) \times (E \otimes \mathbb{Q}_p)$ via the canonical ring isomorphism

$$(17) \quad \begin{aligned} K \otimes \mathbb{Q}_p &\xrightarrow{\sim} (E \otimes \mathbb{Q}_p) \times (E \otimes \mathbb{Q}_p) \\ x &\mapsto (\phi_\Sigma(x), \phi_\Sigma(\bar{x})) = ((\sigma(x))_{\sigma \in \Sigma}, (\sigma(\bar{x}))_{\sigma \in \Sigma}) \end{aligned}$$

Conventions for certain p -adic maps. The isomorphism (17) induces an identification of $V \otimes \mathbb{Q}_p$ with a module over $(E \otimes \mathbb{Q}_p) \times (E \otimes \mathbb{Q}_p)$. Similarly, the isomorphism (17) identifies modules over $\mathcal{O}_K \otimes \mathbb{Z}_p$ with modules over $(\mathcal{O}_E \otimes \mathbb{Z}_p) \times (\mathcal{O}_E \otimes \mathbb{Z}_p)$. Given continuous maps (most often characters)

$$\chi_1, \chi_2 : \text{GL}_n(E \otimes \mathbb{Q}_p) \rightarrow \mathbb{C}_p^\times,$$

we write (χ_1, χ_2^{-1}) to mean the function on $\text{GL}_n(K \otimes \mathbb{Q}_p)$ defined by

$$(18) \quad a \mapsto \chi_1(\phi_\Sigma(a)) \chi_2(\phi_\Sigma({}^t \bar{a}^{-1})),$$

so

$${}^t \bar{a}^{-1} \mapsto \chi_1(\phi_\Sigma({}^t \bar{a}^{-1})) \chi_2(\phi_\Sigma(a)).$$

Hermitian matrices. In particular, if $\beta \in \text{GL}_n(\mathcal{O}_K \otimes \mathbb{Z}_p)$ is Hermitian, then

$$\beta \mapsto \chi_1(\phi_\Sigma(\beta)) \chi_2(\phi_\Sigma(\beta^{-1})).$$

We identify the space of Hermitian matrices with its image under the map induced by ϕ_Σ (i.e project onto the first factor in the map (17)).

The unitary groups $U(V)$ and $U(W)$ at p . Let $H = U(V)$ and $G = U(W) = U(2V)$ (as in Section 2.1). As in Section 2.1, fix a basis

e_1, \dots, e_n for V over K . We identify V^d and V_d with V via the isomorphisms

$$\begin{aligned} V^d &\xrightarrow{\sim} V \xrightarrow{\sim} V_d \\ (e_i, e_i) &\mapsto e_i \mapsto (e_i, -e_i) \end{aligned}$$

induced by the choice of basis e_1, \dots, e_n . Via the isomorphism (17), we identify $V \otimes \mathbb{Q}_p$ with the direct product of two n -dimensional vector spaces over $E \otimes \mathbb{Q}_p$, each of which is fixed under the action of the unitary group $H(E \otimes \mathbb{Q}_p)$ and which are swapped under the involution induced by complex conjugation ϵ on K . Similarly, we identify $W \otimes \mathbb{Q}_p$ with the direct product of two $2n$ -dimensional vector spaces over $E \otimes \mathbb{Q}_p$, each of which is fixed under the action of the unitary group $G(E \otimes \mathbb{Q}_p)$ and which are swapped under the involution induced by ϵ .

For each prime $v \in \Sigma$, we use the following identifications, induced by the projection map ϕ_Σ defined in (16):

$$\begin{aligned} H(E_v) &\xrightarrow{\sim} \mathrm{GL}_{K_v}(V \otimes K_v) \\ G(E_v) &\xrightarrow{\sim} \mathrm{GL}_{K_v}(W \otimes K_v). \end{aligned}$$

For each prime $v \in \Sigma$, the choice of basis e_1, \dots, e_n together with the above isomorphism also induces identifications

$$\begin{aligned} H(E_v) &\xrightarrow{\sim} \mathrm{GL}_n(K_v) \xrightarrow{\sim} \mathrm{GL}_n(E_v) \\ (19) \quad G(E_v) &\xrightarrow{\sim} \mathrm{GL}_{2n}(K_v) \xrightarrow{\sim} \mathrm{GL}_{2n}(E_v). \end{aligned}$$

The parabolic and Levi subgroups at p . Let v be a prime in Σ . The image of $M(E_v)$ in $\mathrm{GL}_{2n}(E_v) \xrightarrow{\sim} G(E_v)$ under the above identifications is

$$\left\{ \mathrm{diag}(B, A) \in \mathrm{GL}_{2n}(E_v) \mid B, A \in \mathrm{GL}_n(E_v) \xrightarrow{\sim} \mathrm{GL}_{K_v}(V \otimes_K K_v) \right\} \xrightarrow{\sim} \mathrm{GL}_n(E_v) \times \mathrm{GL}_n(E_v).$$

The isomorphism (2) together with the isomorphism (17) gives identifications

$$\begin{aligned} M(E_v) &\xrightarrow{\sim} \mathrm{GL}_{K_v}(V \otimes_K K_v) \times \mathrm{GL}_{K_{\bar{v}}}(V \otimes_K K_{\bar{v}}) \\ (20) \quad &\xrightarrow{\sim} \mathrm{GL}_{E_v}(V \otimes E_v) \times \mathrm{GL}_{E_v}(V \otimes_E E_v). \end{aligned}$$

Let $m = \mathrm{diag}(A, B) \in \mathrm{GL}_{2n}(E_v)$ be an element of the Levi subgroup $M(E_v)$. The image of m under the isomorphism (20) is $(A, {}^t B^{-1})$.

The image of the parabolic subgroup $P(E \otimes \mathbb{Q}_p)$ in $\mathrm{GL}_{2n}(E_v)$ under the identification (19) of $G(E \otimes \mathbb{Q}_p)$ with $\mathrm{GL}_{2n}(E_v)$ is

$$\left\{ \left(\begin{pmatrix} B & C \\ 0 & A \end{pmatrix} \right) \mid A, B \in \mathrm{GL}_n(E_v), C \in M_{n \times n}(E_v) \right\}.$$

Let α be an element of $P(E \otimes \mathbb{Q}_p)$ whose image in $\mathrm{GL}_{2n}(E \otimes \mathbb{Q}_p)$ under the above identification is $\begin{pmatrix} B & C \\ 0 & A \end{pmatrix}$. Let χ_p denote the restriction of the Hecke character χ to

$$K \otimes \mathbb{Q}_p \stackrel{(17)}{=} \prod_{v \in \Sigma} K_v \times \prod_{v \in \Sigma} K_{\bar{v}} = \prod_{v \in \Sigma} E_v \times \prod_{v \in \Sigma} E_{\bar{v}}.$$

Writing

$$\begin{aligned} \chi_p &= (\chi_1, \chi_2^{-1}) \\ |\cdot|_p^{-s} &= \prod_{v|p} |\cdot|_v^{-s} \end{aligned}$$

as in (18) and evaluating $\chi_p \cdot |\cdot|_p^{-s}$ on α as in Section 2.2.2, we have that

$$\chi_p(\alpha) |\alpha|_p^{-s} = \chi_1(\det A) \chi_2(\det B) \prod_{v \in \mathcal{O}_E, v|p} |\det(B^{-1}A)|_v^{-s}.$$

2.2.8. Siegel sections at p . We modify the Siegel section in [HLS06], because it does not preserve certain information about determinants that is necessary for the interpolation of special values of non-holomorphic Eisenstein series. Since different sections at p are likely best in different contexts, we also give a recipe for constructing certain kinds of sections at p that can be used to produce an Eisenstein measure in Section 4. At the end of this section, we discuss some natural generalizations to our construction.

Given the character $\chi_p := \prod_{v|p} \chi_v$ of $(K \otimes \mathbb{Q}_p)^\times \cong (E \otimes \mathbb{Q}_p)^\times \times (E \otimes \mathbb{Q}_p)^\times$ (where the isomorphism is as in (17)), we write $\chi_p = (\chi_1, \chi_2^{-1})$, using the convention established in Section 2.2.7.

Let $v \subseteq K$ be a prime in Σ . For any K -vector space U , denote by U_v the vector space $U \otimes_K K_v$. To each Schwartz function

$$\Phi_v : \mathrm{Hom}_{K_v}(V_v, V_{v,d} \oplus V_v^d) \rightarrow K_v,$$

we attach a section

$$f^{\Phi_v} \in \mathbf{Ind}_{P(E_v)}^{G(E_v)} (\chi_v \cdot |\cdot|_v^{-2s})$$

as follows. Consider the decomposition

$$\begin{aligned} \mathrm{Hom}_{K_v}(V_v, W_v) &= \mathrm{Hom}_{K_v}(V_v, V_{d,v}) \oplus \mathrm{Hom}_{K_v}(V_v, V_v^d) \\ X &= (X_1, X_2). \end{aligned}$$

Let

$$\begin{aligned} \mathbf{X} &= \{X \in \mathrm{Hom}_{K_v}(V_v, W_v) \mid X(V_v) = V_v^d\} \\ &= \{(0, X) \mid X : V_v \rightarrow V_v^d \text{ is an isomorphism}\}. \end{aligned}$$

Note that for any $X \in \mathbf{X}$, the composition $V_v \xrightarrow{X} V_v^d \xrightarrow{\sim} V_v$ is an isomorphism of V with itself. So we may identify \mathbf{X} with $\text{Aut}_{K_v}(V_v)$. Via the conventions established in Section 2.2.7, we identify each $g \in G(E_v)$ with its image in $\text{GL}_{K_v}(W_v) \xrightarrow{\sim} \text{GL}_{2n}(K_v) \rightarrow \text{GL}_{2n}(E_v)$. We define the section $f^{\Phi_v} \in \mathbf{Ind}_{P(E_v)}^{G(E_v)}(\chi_v \cdot |\cdot|_v^{-2s})$ by¹

$$(21) \quad f^{\Phi_v}(g) := \chi_{2,v}(\det g) |\det g|_v^{2s} \int_{\mathbf{X}} \Phi(Xg) \chi_{1,v}^{-1} \chi_{2,v}(\det X) |\det X|_v^{4s} d^\times X.$$

It is a simple computation to check that, indeed, $f^{\Phi_v} \in \mathbf{Ind}_{P(E_v)}^{G(E_v)}(\chi_v \cdot |\cdot|_v^{-2s})$. **Fourier expansion at p .** Our choice of a section well-suited to p -adic interpolation relies upon Lemma 10, which we state below.

Given a Schwartz function

$$\begin{aligned} F : \text{Hom}_{K_v}(V_v, V_{d,v}) \oplus \text{Hom}_{K_v}(V_v, V_v^d) &\rightarrow K_v \\ (X_1, X_2) &\mapsto F(X_1, X_2), \end{aligned}$$

we denote by $PF(X_1, X_2)$ the Fourier transform of F in the second variable, i.e.

$$PF(X_1, X_2) = \int_{M_n(E_v)} F(X_1, Y) \mathbf{e}_p^{\text{tr}(-Y \cdot {}^t X_2)} dY.$$

Note that

$$PPF(X_1, X_2) = F(X_1, -X_2).$$

Lemma 10. ² *Let Γ be a compact subgroup of $\text{GL}_n(\mathcal{O}_{E_v})$, and let F be a locally constant Schwartz function*

$$\begin{aligned} F : \text{Hom}_{K_v}(V_v, V_{d,v}) \oplus \text{Hom}_{K_v}(V_v, V_v^d) &\rightarrow K_v \\ (X_1, X_2) &\mapsto F(X_1, X_2) \end{aligned}$$

whose support in the first variable is Γ and such that

$$F(X, {}^t X^{-1} Y) = \chi_1 \chi_2^{-1}(\det(X)) F(1, Y)$$

for all X in Γ and Y in $M_n(E_v)$. Then the Fourier coefficient

$$c(\beta, 1; f^{PF(-X, Y)}) =: f_\beta^{PF(-X, Y)}$$

¹A typo in the exponents in Equation (21) appeared in the first three versions of this paper on the arxiv. The correct expression appears here.

²There were three typos in Lemma 10 (all concerning volume(Γ)) that appeared in the first three versions of this paper on the arxiv. The correct statement and proof appears here.

of $f^{PF(-X,Y)}$ at $\beta \in M_n(E_v)$ is

$$c(\beta, 1; f^{PF(-X,Y)}) = f_\beta^{PF(-X,Y)}(1) = \text{volume}(\Gamma) \cdot F(1, {}^t\beta).$$

Proof. Throughout the proof, we drop the subscript v from Φ (to simplify the notation). For any Schwartz function³

$$\Phi : \text{Hom}_{K_v}(V_v, V_{d,v}) \oplus \text{Hom}_{K_v}(V_v, V_v^d) \rightarrow K_v,$$

$$\begin{aligned} f_\beta^\Phi(1) &= \int_{M_n(E_v)} f^\Phi \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \right) \mathbf{e}_p(-\text{tr } \beta N) dN \\ &= \int_{GL_n(E_v)} \chi_1^{-1} \chi_2(\det X) |\det X|^{4s} \int_{M_n(E_v)} \Phi(-X, -XN) \mathbf{e}^{(-\text{tr}(\beta N))} dN d^\times X \\ &= \int_{GL_n(E_v)} \chi_1^{-1} \chi_2(\det X) |\det X|^{4s} \int_{M_n(E_v)} \Phi(-X, N) \mathbf{e}^{(\text{tr}(\beta X^{-1}N))} dN d^\times X \\ &= \int_{GL_n(E_v)} \chi_1^{-1} \chi_2(\det X) |\det X|^{4s} P\Phi(-X, -\beta X^{-1}) d^\times X. \end{aligned}$$

So

$$\begin{aligned} f_\beta^{PF(-X,Y)}(1) &= \int_{GL_n(E_v)} \chi_1^{-1} \chi_2(\det X) |\det X|^{4s} PPF(X, -{}^tX^{-1}{}^t\beta) d^\times X \\ &= \int_{GL_n(E_v)} \chi_1^{-1} \chi_2(\det X) |\det X|^{4s} F(X, {}^tX^{-1}{}^t\beta) d^\times X \\ &= \int_\Gamma F(1, {}^t\beta) d^\times X \\ &= \text{volume}(\Gamma) \cdot F(1, {}^t\beta). \end{aligned}$$

■

A choice of a Schwartz function. We now choose specific Schwartz functions F meeting the criteria of Lemma 10. These Siegel sections play a key role in our construction of the Eisenstein measure.

A *parahoric subgroup* of $GL_n(E_v)$. Let $(a, b) = ((a(\sigma), b(\sigma))_{\sigma \in \Sigma})$ be the signature of $\langle v_1, v_2 \rangle_V$. For each $v \in \Sigma$, let

$$a_v = n_{1,v} + \cdots + n_{t(v),v}$$

$$b_v = n_{t(v)+1,v} + \cdots + n_{r(v),v}$$

be partitions of $a(\sigma_v)$ and $b(\sigma_v)$.

Let Q_v be the standard parabolic of $GL_n(\mathcal{O}_{E_v})$ corresponding to the partition $n = n_{1,v} + \cdots + n_{r(v),v}$. Write

$$Q_v = L_v U_v$$

for the decomposition of Q_v into its Levi L_v and its upper triangular unipotent U_v . (So L_v has non-singular $n_{i,v} \times n_{i,v}$ blocks running down the diagonal and zeroes everywhere else.) Given $m \in L_v$, we write

³A typo in the exponent of $|\det X|$ the proof of Lemma 10 appeared in the first three versions of this paper on the arxiv.

$m = (m_1, \dots, m_r)$, where each m_i is an $n_{i,v} \times n_{i,v}$ matrix in $GL_n(\mathcal{O}_{E_v})$. Let I_v be the parahoric subgroup corresponding to Q_v (i.e. the terms below the $n_{i,v} \times n_{i,v}$ -blocks along the diagonal are in $\mathfrak{p}_v \mathcal{O}_{E_v}$, where \mathfrak{p}_v is the prime over p corresponding to v), and write

$$I_v = N_v L_v U_v,$$

where N_v is the lower triangular unipotent for the parahoric.

A convenient choice of Schwartz functions supported on the parahoric.

For each $v \in \Sigma$, let $\mu_{1,v}, \dots, \mu_{r(v),v}$ be characters of $\mathcal{O}_{E_v}^\times$, and let $\mu_v = (\mu_{1,v}, \dots, \mu_{r(v),v})$. View each character $\mu_{i,v}$ as a character of $GL_{n_i}(\mathcal{O}_{E_v})$ via composition with the determinant.

For $nm u \in I$ (with n, m, u in the lower unipotent radical, Levi, and upper unipotent radical, respectively), we define

$$\phi_{\mu,v}(nm u) = \prod_{v \in \Sigma} \mu_{1,v}(m_{1,v}) \cdots \mu_{r(v),v}(m_{r(v),v});$$

Note that if each $\mu_{i,v}$ is defined by $\mu_{i,v}(x) = x^d$ for some integer d , then

$$\phi_{\mu,v}(nm u) = \det(m_1)^d \cdots \det(m_{r(v)})^d = \det(nm u)^d.$$

We extend the function $\phi_{\mu,v}$ to all of $M_n(E_v)$ by defining

$$\phi_{\mu,v}(X) = 0$$

for all X not in I_v .

Let $c \geq \max_{v \in \Sigma, 1 \leq i \leq r} (1, \text{ord}_p(\text{conductor}(\mu_{i,v})))$, and let $\Gamma = \Gamma(c) = \prod_{v \in \Sigma} \Gamma_v(c)$ be the subgroup of $I = \prod_{v \in \Sigma} I_v$ whose terms below the $n_{i,v} \times n_{i,v}$ -blocks along the diagonal are in $\mathfrak{p}_v^c \mathcal{O}_{E_v}$ for all v .

Let Φ_1 be the function of $M_{n \times n}(\mathcal{O}_E)$ supported in $\Gamma_v(c)$ (and extended by 0 to all of $M_{n \times n}(\mathcal{O}_{E_v})$) such that

$$\Phi_{1,v}(x) = \phi_{\mu,v}(x),$$

for all $x \in \Gamma_v(c)$.

Define⁴

$$(22) \quad \Phi_{\chi_v, \mu_v}(X_1, X_2) = \text{volume}(\Gamma_v)^{-1} \Phi_{1,v}(-X_1) \cdot \hat{\phi}_{\nu,v}(X_2),$$

where

$$\nu_v = \chi_{1,v}^{-1} \cdot \chi_{2,v} \cdot \mu_v$$

and $\hat{\phi}_{\nu,v}$ denotes the Fourier expansion of $\phi_{\nu,v}$ (so that Φ_{χ_v, μ_v} is a partial Fourier expansion in the second variable in the sense of Lemma 10). We define

$$(23) \quad f^{\chi, \mu_v} = f^{\Phi_{\chi, \mu_v}}.$$

⁴Prior versions of this paper had $2X_2$ instead of X_2 on the right hand side of Equation (22). That was a typo.

By Lemma 10, the β -th Fourier coefficient of f^{χ, μ_v} is

$$(24) \quad f_\beta^{\chi, \mu_v} = \phi_{\nu, v}({}^t\beta).$$

Observe that $f_\beta^{\chi, \mu_v} = 0$ whenever β is of rank less than n and is, in fact, 0 whenever $\beta \notin \Gamma$. Also, note that the above definitions and discussion still hold if we replace each character μ_v by a locally constant function.

We write $\mu = \prod_{v \in \sigma} \mu_v$, $\nu = \prod_{v \in \sigma} \nu_v$, $\phi_\mu = \prod_{v \in \Sigma} (\phi_{\mu, v})$, and $f^{\chi, \mu} = \otimes_{v \in \Sigma} f^{\chi, \mu_v}$.

Remark 11. In this remark and the following remark, we suppress the subscript v where it is clear that we are working at the place v . Let δ be an $n \times n$ matrix, and let γ be an invertible $n \times n$ matrix with entries in \mathcal{O}_E . Note that one can modify the definition of the local sections given in Equation (21) so that the integral is again over $\mathrm{GL}_n(\mathcal{O}_{E_v})$ but Φ is evaluated at $(X(\gamma C \gamma^{-1}), X(C\delta + D))$ for each $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Then we can define Φ so that we obtain a different section from the one given above, but with similar Fourier coefficients to the ones given above (the main difference being the support of the Fourier coefficients). There are also other relatively minor modifications one can make to this construction to obtain other local Siegel sections at p with Fourier coefficients of a similar form.

Remark 12. The viewpoint of highest weight theory (see, e.g., [Shi00, Section 12]) suggests there is a more natural way to define ϕ_ν , namely in terms of leading principal minors. Indeed, Christopher Skinner points out that, from that perspective, it is more natural to replace ϕ_ν with the function defined as follows.

Denote by \mathfrak{X} the subset of $M_n(\mathcal{O}_{E_v})$ consisting of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with $A \in M_{a_v}(\mathcal{O}_{E_v})$, such that the determinant of each of the leading principal $n_{1,v} + \dots + n_{i,v}$ -th minors of A is in $\mathcal{O}_{E_v}^\times$ for $i = 1, \dots, t(v)$, and the determinant of each of the leading principal $n_{t(v)+1,v} + \dots + n_{i,v}$ -th minors of D is in $\mathcal{O}_{E_v}^\times$ for $i = t(v) + 1, \dots, r(v)$. Denote by A_i the determinant of the leading principal i -th minor.

Now replace ϕ_ν by the function that is supported on \mathfrak{X} and is defined

for $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{X}$ by

$$\phi_\nu(X) = \prod_{i=1}^{t(v)-1} (\nu_i \cdot \nu_{i+1,v}^{-1})(A_{n_{1,v} + \dots + n_{i,v}}) \cdot \nu_{t(v),v}(A) \prod_{i=t(v)+1}^{r(v)-1} (\nu_i \cdot \nu_{i+1,v}^{-1})(D_{n_{t(v)+1,v} + \dots + n_{i,v}}) \cdot \nu_{r(v),v}(D)$$

This new definition of ϕ_ν agrees with the prior definition of ϕ_ν on Γ but has larger support. The Fourier coefficients still have the same form; the only change is that they are supported on a larger group. Either choice of ϕ_ν is fine for the application of the differential operators and construction of the Eisenstein measure in the current paper. For [Eis], however, it is more natural to use this latter definition of ϕ_ν .

2.2.9. The sections at finite places of E not dividing p or ∞ . Let \mathfrak{b} be an ideal in \mathcal{O}_E prime to p . For each finite place v prime to p , [Shi97, Section 18] explains how to define sections $f_v^\mathfrak{b} = f_v^\mathfrak{b}(\bullet; \chi_v, s) \in \mathbf{Ind}_{P(E_v)}^{G(E_v)}(\chi_v, s)$ with the following property: By [Shi97, Proposition 19.2], whenever the Fourier coefficient $c(\beta, m(1); f_v^\mathfrak{b})$ is non-zero,

(25)

$$\prod_{v \nmid p\infty} c(\beta, m(1); f_v^\mathfrak{b}) = N_{E/\mathbb{Q}}(\mathfrak{b}\mathcal{O}_E)^{-n^2} \prod_{i=0}^{n-1} L^p(2s - i, \chi_E^{-1}\tau^i)^{-1} \prod_{v \nmid p\infty} P_{\beta, v, \mathfrak{b}}(\chi_E(\pi_v)^{-1} |\pi_v|_v^{2s}),$$

where:

- (1) the product is over primes of E ;
- (2) the Hecke character χ_E is the restriction of χ to E ;
- (3) the function $P_{\beta, v, \mathfrak{b}}$ is a polynomial that is dependent only on β , v , and \mathfrak{b} and has coefficients in \mathbb{Z} and constant term 1;
- (4) the polynomial $P_{\beta, v, \mathfrak{b}}$ is identically 1 for all but finitely many v ,
- (5) τ is the Hecke character of E corresponding to K/E ,
- (6) π_v is a uniformizer of $\mathcal{O}_{E, v}$, viewed as an element of K^\times prime to p . and
- (7)

$$L^p(r, \chi_F^{-1}\tau^i) = \prod_{v \nmid p\infty \text{ cond } \tau} (1 - \chi_v(\pi_v)^{-1} \tau^i(\pi_v) |\pi_v|_v^r)^{-1}.$$

Note that only the factor $\prod_{v \nmid p\infty} P_{\beta, v}(\chi_F(\pi_v)^{-1} |\pi_v|_v^{2s})$ depends on β .

Remark 13. The above sections away from p , as constructed in [Shi97], are built from characteristic functions of lattices (which one can choose to have certain properties corresponding to the choice of ideal \mathfrak{b} and the desired level of the Eisenstein series away from p). These are the same as in [HLS06].

2.2.10. Global Fourier coefficients. Recall that by Equation (9), the Fourier coefficients $c(\beta, h; f)$ are completely determined by the coefficients $c(\beta, 1_n; f)$. In Proposition 14, we combine the results of Sections 2.2.6, 2.2.8, and 2.2.9 in order to give the global Fourier coefficients of the Eisenstein series E_f .

Let χ be a unitary Hecke character meeting the conditions of Section 2.2.1, and furthermore, suppose the infinity type of χ is

$$(26) \quad 0 \cdot \sum_{\sigma \in \Sigma} \sigma - \sum_{\sigma \in \Sigma} \left(\frac{k}{2} + \nu(\sigma) \right) (\sigma - \bar{\sigma})$$

(i.e. $k(\sigma) = k \in \mathbb{Z}$ for all $\sigma \in \Sigma$). Let $c(n, K)$ be the constant dependent only upon n and K defined in Equation (10).

Proposition 14. *Let*

$$(27) \quad f_{k,\nu,\mu,\chi} := f_{k,\nu,\mu,\chi,\mathfrak{b}} := f^{\chi,\mu} \otimes f_{\infty}^{k,\nu} \left(\bullet; i1_n, \chi, \frac{k}{2} \right) \otimes f^{\mathfrak{b}} \in \mathbf{Ind}_{P(\mathbb{A}_E)}^{G(\mathbb{A}_E)} \left(\chi \cdot |\cdot|_K^{-\frac{k}{2}} \right),$$

with χ as in Equation (26), $f^{\chi,\mu}$ the section at p defined in Equation (23), $f_{\infty}^{k,\nu}$ the section at ∞ defined in Section 2.2.6, and $f^{\mathfrak{b}}$ the section away from p and ∞ defined in Section 2.2.9.

Then all the nonzero Fourier coefficients $c(\beta, 1_n; f_{k,\nu,\mu,\chi})$ are given by

$$(28) \quad D(n, K, \mathfrak{b}, p, k) \prod_{v \nmid p\infty} P_{\beta,v,\mathfrak{b}} \left(\chi_E(\pi_v)^{-1} |\pi_v|_v^k \right) \phi_{(\chi_1^{-1} \chi_2 \mu)}({}^t \beta) \prod_{v \in \Sigma} \sigma_v(\det \beta)^{k-n} \mathbf{e}(\mathrm{itr}_{E/\mathbb{Q}}(\beta)).$$

where

$$D(n, K, \mathfrak{b}, p, k) = C(n, K) N(\mathfrak{b} \mathcal{O}_E)^{-n^2} \prod_{i=0}^{n-1} \left(2^{(1-n)n} i^{-nk} (2\pi)^{nk} \left(\pi^{n(n-1)/2} \prod_{t=0}^{n-1} \Gamma(k-t) \right)^{-1} \right)^{[E:\mathbb{Q}]} \prod_{i=0}^{n-1} L^p(k-i, \chi_E^{-1} \tau^i)^{-1}.$$

Proof. This follows directly from Section 2.2.4, Lemma 10, and Equations (24), (25), and (15). \blacksquare

2.2.11. q -expansions. Let $f = \otimes_v f_v \in \otimes_v I(\chi_v, s)$ be a section whose component at ∞ is $f_{\infty}^{k,\nu}$, defined as in Section 2.2.6. For $z = x + iy \in \mathcal{H}_n$, $h \in GL_n(\mathbb{A}_{K,f})$, and $\alpha \in \prod_{v|\infty} G_v$ such that $\alpha \mathfrak{I} = z$, we put

$$E_{k,\nu}(z; h, \chi, \mu, s) := j_{\alpha}^{(k(\sigma)_{\sigma}, \nu(\sigma)_{\sigma})}(\mathfrak{I}) E_f(m(h)\alpha).$$

Then, as explained in [Shi97, Lemma 18.7(2)], one can give the Fourier expansion for the function $E_f(z; h, k, \nu, s)$ of the variable z in terms of the Fourier coefficients given above. (Note that Equation (14) gives the correction factor for using $\mathfrak{I} \in \mathcal{H}_n$ instead of i .)

In particular, when the infinity type of χ is as in (26) and f is the Siegel section defined in Equation (27), $E_{k,\nu}(z; h, \chi, \mu, \frac{k}{2})$ is a holomorphic function of z such that

$$D(n, K, \mathfrak{b}, p, k)^{-1} E_{k,\nu} \left(z; 1, \chi, k, \nu, \frac{k}{2} \right) = \sum_{\beta} \prod_{v \nmid p\infty} P_{\beta, v, \mathfrak{b}} \left(\chi_E(\pi_v)^{-1} |\pi_v|_v^k \right) \phi_{(\chi_1^{-1} \chi_2 \mu)}({}^t \beta) \prod_{v \in \Sigma} \sigma_v(\det \beta)^{k-n} \mathbf{e}_{\infty}(\mathrm{tr}(\beta z)).$$

(The above sum is over $\beta \in \mathrm{Her}_n$ such that the Fourier coefficient at β is nonzero.) Observe that for this choice of f , the Fourier coefficients of $E_f(z; h)$ are independent of z and algebraic and p -integral. Let \mathcal{K} be the compact subgroup of $G(\mathbb{A})$ so that we may view $E_{k,\nu}$ as an automorphic form on the complex points of ${}_{\mathcal{K}}Sh(W)$ (i.e. \mathcal{K} corresponds to the above choice of local Siegel sections).

Denote by $G_{k,\nu,\chi,\mu}$ the automorphic form on the ${}_{\mathcal{K}}Sh(W)$ whose algebraically defined q -expansion (i.e. its value at the Mumford object, generalizing the Tate curve; see [Lan08, Lan11, Har86] for more on algebraic q -expansions) at a cusp L is

$$(29) \quad G_{k,\nu,\chi,\mu}(q) = \sum_{\beta \in L} \left(\alpha(\beta, \chi) \phi_{(\chi_1^{-1} \chi_2 \mu)}({}^t \beta) \prod_{v \in \Sigma} \sigma_v(\det \beta)^{k-n} \right) q^{\beta},$$

where

$$\alpha(\beta, \chi) = \prod_{v \nmid p\infty} P_{\beta, v, \mathfrak{b}} \left(\chi_E(\pi_v)^{-1} |\pi_v|_v^k \right),$$

$k > 2n - 1$, and χ is a Hecke character meeting the conditions of Section 2.2. Note that the q -expansion coefficients of $G_{k,\nu,\chi,\mu}(q)$ are the same as those in the Fourier expansion of the holomorphic function $D(n, K, \mathfrak{b}, p, k)^{-1} E_{k,\nu}(z; 1, \chi, \mu, \frac{k}{2})$ given above, and the value of $G_{k,\nu,\chi,\mu}$ at a \mathbb{C} -valued point of ${}_{\mathcal{K}}Sh(W)$ is the same as the value of $E_{k,\nu}$ at the corresponding point z .

An argument similar to [Kat78, Theorem (3.4.1)] shows that $G_{k,\nu,\chi,\mu}(q)$ is, in fact, the q -expansion of a p -adic automorphic form (simply the image under the canonical map from the space of automorphic forms over $\mathcal{O}_{\mathbb{C}_p}$ to the space of p -adic automorphic forms), which we also denote by $G_{k,\nu,\chi,\mu}$.

Remark 15. Relationship with Katz's Eisenstein series for CM-fields

Plugging $n = 1$ into Equation (28) and comparing with [Kat78, Equations (3.2.6)-(3.2.8)] shows that for $n = 1$, our Eisenstein series are essentially those in [Kat78], and for $n > 1$, our Eisenstein series (and the Fourier coefficients in Equation (28)) are a natural generalization of those in [Kat78].

In particular, since the terms in Equation (28) look more complicated than in [Kat78], we note that due to cancellation when $n = 1$, we obtain

$$c(1, K) = |D_E|^{-1/2},$$

$$D(1, K, \mathfrak{b}, p, k) = |D_E|^{-1/2} N(\mathfrak{b}) \left((-2\pi i)^k \Gamma(k)^{-1} \right)^{[E:\mathbb{Q}]} L^p(k, \chi_E^{-1})^{-1}.$$

2.2.12. Certain p -adic Eisenstein series. Via the isomorphism (17), we identify $\mathcal{O}_K \otimes \mathbb{Z}_p$ with $(\mathcal{O}_E \otimes \mathbb{Z}_p) \times (\mathcal{O}_E \otimes \mathbb{Z}_p)$. For each $v \in \Sigma$, let $r_v = r(v)$ be a positive integer, and let $r = (r_v)_v$. Let

$$(30) \quad T(r) = \underbrace{\prod_{v \in \Sigma} \mathcal{O}_{E_v}^\times \times \cdots \times \mathcal{O}_{E_v}^\times}_{r_v \text{ copies}}.$$

Let μ be a locally constant function on $T(r)$, extended by 0 to all of $\prod_{v \in \Sigma} \underbrace{\mathcal{O}_{E_v} \times \cdots \times \mathcal{O}_{E_v}}_{r_v \text{ copies}}$. Fix integers k and $\nu(\sigma)$, and let $\nu = (\nu(\sigma))_\sigma$.

Let F be a locally constant function on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times$, extended by 0 to all of $\mathcal{O}_K \otimes \mathbb{Z}_p$, such that

$$(31) \quad F(e) = \prod_{\sigma \in \Sigma} \left(\frac{\bar{\sigma}(e)}{\sigma(e)} \right)^{-(\frac{k}{2} + \nu(\sigma))}$$

for all $e \in \mathcal{O}_K^\times$. Then there is an integer d and Hecke characters ξ_1, \dots, ξ_d of K of conductor dividing p^∞ and infinity type $\prod_{\sigma \in \Sigma} \left(\frac{\bar{\sigma}}{\sigma} \right)^{(\frac{k}{2} + \nu(\sigma))}$ such that

$$F(a) = F_1(a) + \cdots + F_d(a),$$

$$F_i = \prod_{v|p} \xi_{i,v}$$

for all $a \in (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times$. Let

$$\psi_{k,\nu} = \prod_{\sigma \in \Sigma} \sigma^{-k} \left(\frac{\bar{\sigma}}{\sigma} \right)^{\nu(\sigma)}.$$

Lemma 16. *Given such a locally constant function F , there is an automorphic form $E_{F,\mu}$ defined over \mathcal{O}_K of weight k, ν , whose q -expansion is given by*

$$E_{F,\mu}(q) = \sum_{i=1}^d E_{F_i,\mu}(q)$$

$$E_{F_i,\mu}(q) = \sum_{\beta \in L} \left(\left(\prod_{v \nmid p^\infty} P_{\beta,v,\mathfrak{b}}(F_i(\pi_v) \psi_{k,\nu}(\pi_v)) \right) \cdot F_i(\det \beta^{-1}) \psi_{k,\nu}(\det \beta^{-1}) \prod_{v \in \Sigma} \sigma_v(\det \beta)^{-n} \phi_\mu(\beta) \right) q^\beta.$$

Proof. We take $E_{F_i,\mu}$ to be the automorphic form $G_{k,\nu,\xi_i,\mu}$ in Equation (29). The lemma then follows immediately. \blacksquare

Remark 17. Similarly to the discussion in [Kat78, Section 3], we may generalize the statement of Lemma 16 to the case of locally constant functions F and μ having image in any \mathcal{O}_K -algebra R (in which case $E_{F,\mu}$ is an automorphic form over R).

Suppose now that F is a continuous function satisfying Equation (31) supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times$ and that μ is a continuous function on $T(r)$ extended by 0 to all of $\underbrace{\prod_{v \in \Sigma} \mathcal{O}_{E_v} \times \cdots \times \mathcal{O}_{E_v}}_{r_v \text{ copies}}$. Then there is a

p -adic automorphic form $E_{F,\mu}$ obtained via the q -expansion principle and an argument similar to the proof of [Kat78, Theorem (3.4.1)]. The q -expansion coefficients of $E_{F,\mu}$ are essentially p -adic limits of q -expansion coefficients of automorphic forms E_{F_i, μ_i} with F_i, μ_i locally constant and F_i satisfying Equation (31). When μ and F are locally constant, this automorphic form is defined over \mathcal{O}_K and is just the image of the automorphic form from Lemma 16 under the canonical map from the space $\mathcal{A}(k, \nu, \mu, \mathcal{O}_K)$ of weight k, ν automorphic forms on ${}_{\mathcal{K}}Sh(W)$ into the space \mathcal{V} of p -adic automorphic forms (viewed as sections on the Igusa tower).

Thus, it follows that for *any* continuous functions F supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times$ satisfying Equation (31) (for *some* integers k and ν) and μ supported on $\underbrace{\prod_{v \in \Sigma} \mathcal{O}_{E_v}^\times \times \cdots \times \mathcal{O}_{E_v}^\times}_{r_v \text{ copies}}$ with values in an \mathcal{O}_K -algebra R ,

we obtain a p -adic automorphic form $E_{F,\mu}$ defined over R whose q -expansion coefficients are limits of q -expansion coefficients of some automorphic forms E_{F_i, μ_i} for some locally constant F_i 's and μ_i 's.

Remark 18. When F and μ are locally constant $\bar{\mathbb{Q}}$ -valued functions, $E_{F,\mu}$ is just a sum of the analytically defined Eisenstein series discussed earlier.

2.2.13. *Hecke characters of type A_0 , viewed p -adically.* Let

$$\chi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$$

be a Hecke character of type A_0 , so on K ,

$$\chi_\infty(a) = \iota_\infty \circ \left(\prod_{\sigma \in \Sigma} \left(\frac{1}{\sigma} \right)^k \left(\frac{\bar{\sigma}}{\sigma} \right)^{d(\sigma)} \right)$$

with k and $d(\sigma)$ in \mathbb{Z} for all σ . Note that on K ,

$$\iota_p \circ \left(\prod_{\sigma \in \Sigma} \left(\frac{1}{\sigma} \right)^k \left(\frac{\bar{\sigma}}{\sigma} \right)^{d(\sigma)} \right)$$

defines a p -adic character that extends continuously to a (p -adically) continuous $\bar{\mathbb{Q}}_p$ -valued character on $K \otimes \mathbb{Q}_p$. So we can define a p -adic Hecke character on the finite idèles

$$(32) \quad \tilde{\chi} = \prod_{v \nmid \infty} \tilde{\chi}_v : K^\times \backslash \mathbb{A}_K^{\times, \infty} \rightarrow \bar{\mathbb{Q}}_p^\times$$

by

$$\begin{aligned} \tilde{\chi}_v &= \chi_v, \text{ if } v \nmid p \\ \tilde{\chi}_p &= \chi_p \cdot \left(\iota_p \circ \left(\prod_{\sigma \in \Sigma} \left(\frac{1}{\sigma} \right)^k \left(\frac{\bar{\sigma}}{\sigma} \right)^{d(\sigma)} \right) \right). \end{aligned}$$

Note that the restriction of $\tilde{\chi}$ to $\mathbb{A}_K^{\times, p, \infty} \times (\mathcal{O}_K \times \mathbb{Z}_p)^\times$ gives a p -adic character

$$\mathbb{A}_K^{\times, p, \infty} \times (\mathcal{O}_K \times \mathbb{Z}_p)^\times \rightarrow \mathcal{O}_{\mathbb{C}_p}^\times.$$

When we make the shift

$$\chi \rightarrow \tilde{\chi},$$

we shall refer to “shifting to p .”

Example 19. Let $\nu = (\nu(\sigma))_{\sigma \in \Sigma} \in \mathbb{Z}^\Sigma$. Let χ be a Hecke character of type A_0 and whose conductor divides p^∞ and whose infinity-type is

$$\psi_{k, \nu} = \prod_{\sigma \in \Sigma} \sigma^{-k} \left(\frac{\bar{\sigma}}{\sigma} \right)^{\nu(\sigma)}.$$

Let

$$F = \prod_{v \mid p} \tilde{\chi}_v,$$

where $\tilde{\chi}$ is defined as in Equation (32). Then $F(e) = 1$ for all $e \in \mathcal{O}_K^\times$, and

$$(33) \quad E_{F, \mu}(q) = \sum_{\beta} \left(\left(\prod_{v \nmid p^\infty} P_{\beta, v, \mathfrak{b}}(F(\pi_v)) \right) \cdot F(\det \beta^{-1}) N_{E/\mathbb{Q}}(\det \beta)^{-n} \phi_\mu(\beta) \right) q^\beta.$$

If $\chi' = \chi \cdot |\cdot|_{K/\mathbb{Q}}^{\frac{k}{2}}$, then

$$E_{F, \mu} = G_{k, \nu, \chi', \mu}.$$

3. DIFFERENTIAL OPERATORS

Throughout this section, let $k \in \mathbb{Z}$, and let $d = (d(\sigma))_{\sigma \in \Sigma}$, $\nu = (\nu(\sigma))_{\sigma \in \Sigma}$ with $d(\sigma), \nu(\sigma) \in \mathbb{Z}$, $d(\sigma) > 0$, for all σ . We write $k - d, \nu - d$ to denote $(k - d(\sigma))_{\sigma \in \Sigma}, (\nu(\sigma) - d(\sigma))_{\sigma \in \Sigma}$.

We briefly review the most important details of certain differential operators that will allow us to p -adically interpolate special values of the Eisenstein series $E_{k,\nu}$ (normalized by a period), including certain cases where the Eisenstein series is non-holomorphic (for example, $E_{k,\nu}(z; h, \chi, \mu, s)$ with $s \neq \frac{k}{2}$, as well as the case where k is no longer an integer).

Since these differential operators are discussed in thorough detail in [Kat78, Shi84, Shi97, Shi00, Eis12], we use this section of the paper to summarize only the basic properties necessary for this paper.

C^∞ differential operators. There is a differential operator $D_{k,\nu}^d$ that acts on C^∞ -automorphic forms on \mathcal{H}_n and satisfies:

$$D_{k,\nu}^d (j_\alpha^{k,\nu}(z)^{-1}) = \Psi_{n,d,k} \cdot \det(z - {}^t\bar{z})^{-d} \cdot j_\alpha^{k+2d,\nu-d}(z)^{-1} |j_\alpha(z)|^{2d},$$

where

$$\Psi_{n,d,k} = \prod_{\sigma \in \Sigma} \prod_{i=1}^n \prod_{j=1}^{d(\sigma)} (i - j - k).$$

In particular,

(34)

$$D_{k,\nu}^d \left(E_{k,\nu} \left(z; h, \chi, \mu, \frac{k}{2} \right) \right) = (\det(z - {}^t\bar{z})^{-d} \cdot \Psi_{n,d,k}) E_{k+2d,\nu-d} \left(z; h, \chi, \mu, \frac{k}{2} \right)$$

(where $E_{k,\nu}$ is defined as in Section 2.2.11).

The operator $D_{k,\nu}^d$ is a special case of the C^∞ -differential operators (often called “Shimura-Maass” operators) studied extensively by Shimura [Shi84, Shi97, Shi00] (and also studied algebro-geometrically in [Eis12]). It is the operator that would be denoted $D_{\det^{k+\nu} \otimes \det^{-\nu}}^{\det^d}$ in the conventions of [Shi00, Equation (12.17)] (generalized from $SU(\eta)$ to $U(\eta)$, since [Shi00] only deals with special unitary groups) and [Eis12, Section 6].⁵ It is also the operator that would be denoted $D_k(\det^d)$ in the conventions of [Shi97, Lemma 23.5], and it is the generalization to $U(\eta_n)$ of the operator Δ_k^d in [Shi00, Equation (17.20)].

⁵Note that there should be a $2\pi i$ on the right-hand side of [Eis12, Equation (3.29)]. The reason for this is explained at the beginning of [Har81, Section 4.4]. Correspondingly, the C^∞ -differential operator in [Eis12] is actually a power of $2\pi i$ times Shimura’s differential operators discussed in [Shi00, Shi97].

Differential Operators and Pullbacks. As demonstrated in Equation (34), these differential operators change the weight of the automorphic forms to which they are applied. Correspondingly, they also change the weight of the pullback of such an automorphic form to $U(V) \times U(V)$. A precise description of the affect of differential operators on the automorphy factors of pullbacks is given in [Shi97, Sections 23.6-23.9].

p -adic differential operators and their action on q -expansions. By Theorem IX.3 in [Eis12] applied in the special case of scalar-valued automorphic forms, there is a p -adic differential operator θ^d that acts on p -adic automorphic forms (viewed as functions on ordinary abelian varieties with PEL structure corresponding to the choice of Shimura data and a particular moduli problem, i.e. as sections over the corresponding Igusa tower, as discussed in [HLS06] and [Eis12]) whose action on the q -expansion of a p -adic automorphic form f with q -expansion $f(q) = \sum_{\beta} a(\beta)q^{\beta}$ is given by

$$(35) \quad (\theta^d f)(q) = \sum_{\beta} \left(\prod_{\sigma \in \Sigma} \sigma(\det \beta)^{d(\sigma)} \right) a(\beta) q^{\beta}.$$

Relationship between certain values of C^{∞} and p -adic Eisenstein series. Let R be an \mathcal{O}_K -algebra in which p splits completely, together with embeddings

$$\begin{aligned} i_{\infty} : R &\hookrightarrow \mathbb{C} \\ i_p : R &\hookrightarrow R_0 := \varprojlim_n R/p^n R. \end{aligned}$$

As explained in [Kat78] (for Hilbert modular forms) and in [Eis12] (for automorphic forms on $U(n, n)$), the above differential operators can be used to relate certain values of C^{∞} -automorphic forms to values of p -adic automorphic forms. We now apply results from these papers to relate special values of the p -adic and C^{∞} Eisenstein series discussed earlier in this paper.

Denote by L the lattice $\mathcal{O}_K^{2n} \subseteq K^{2n}$. The R -valued points of ${}_{\kappa}Sh(R)$ parametrize tuples \underline{A} consisting of an abelian variety together with a polarization, endomorphism, and level structure (as discussed in [Eis12, Section 2.2]). For each R -valued point of ${}_{\kappa}Sh(R)$ corresponding to an abelian variety \underline{A} , denote by $\omega_{\underline{A}/R}^{\pm}$ an ordered basis over R for the sheaf of differentials $\underline{\omega}_{\underline{A}/R}^{\pm}$ (where $\underline{\omega}_{\underline{A}/R} = \underline{\omega}_{\underline{A}/R}^{+} \oplus \underline{\omega}_{\underline{A}/R}^{-}$ is the usual sheaf of one-forms on \underline{A} over R , as in [Eis12, (2.2)]).

For each point $g \in G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / \mathcal{K}$ (identified with the \mathbb{C} -valued points of ${}_{\kappa}Sh(\mathbb{C})$ as above), let $z = g_{\infty} \mathfrak{z}$, and denote by $\underline{A}(g)$ the tuple

such that $\underline{A}(g)$ is the tuple P_z (dependent upon L) in [Eis12, Section 2.3.2] consisting of a complex abelian variety with PEL structure. Let $\underline{\omega}^\pm(g)$ denote an ordered basis over \mathbb{C} for $\underline{\omega}_{\underline{A}/\mathbb{C}}^\pm$ such that we have an equality of lattices $p_z(L) = L(\underline{A}(g), \underline{\omega}_{\underline{A}/\mathbb{C}}^\pm(g)) \subseteq \mathbb{C}^{2n}$. The precise definitions of P_z , $p_z(L)$, and $L(\underline{A}(g), \underline{\omega}_{\underline{A}/\mathbb{C}}^\pm(g))$ are tedious and unnecessary here; they are discussed in detail in [Eis12, Section 2].

3.0.14. *Comparison of certain values of C^∞ - and p -adic-Eisenstein series.* Now suppose \underline{A} is a tuple over R together with an ordered basis $\underline{\omega}_{\underline{A}/R}^\pm$ over R , such that there is an R -submodule $\text{Split}(\underline{A}/R)$ in $H_{DR}^1(\underline{A}/R)$ inducing a splitting over R

$$\underline{\omega}_{\underline{A}/R} \oplus \text{Split}(\underline{A}/R) \rightarrow H_{DR}^1(\underline{A}/R)$$

giving an isomorphism such that $H_{DR}^1(\underline{A}/R)^\pm \subseteq \underline{\omega}_{\underline{A}/R} \oplus \text{Split}(\underline{\omega}/\underline{A})$, simultaneously satisfying Conditions (\dagger) and (\ddagger) of [Eis12], namely the image of the inclusion $\text{Split}(\underline{A}/R) \otimes_R \mathbb{C} \hookrightarrow H^1(\underline{A}_{\mathbb{C}}^{an}, \mathbb{C})$ is the the anti-holomorphic differentials $H^{0,1}$ and the image of the inclusion $\text{Split}(\underline{A}/R) \otimes_R R_0 \hookrightarrow H^1(\underline{A}_{R_0}^{p\text{-adic}}, R_0)$ is the the unit root subspace of $H^1(\underline{A}_{R_0}^{p\text{-adic}}, R_0)$. (i.e. \underline{A} is a CM abelian variety.)

Now, let $g \in G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/\mathcal{K}$, and suppose that $\underline{A}(g)$ is the extension by scalars to \mathbb{C} of a CM abelian variety \underline{A} over R together with $\underline{\omega}_{\underline{A}/R}^\pm$ and $\text{Split}(\underline{A}/R)$ satisfying Conditions (\dagger) and (\ddagger) (explained in the previous paragraph). Let $\Omega^\pm \in \text{GL}_n(\mathbb{C})^\Sigma$ satisfy

$$\underline{\omega}_{\underline{A}/R}^\pm = \Omega^\pm \cdot \underline{\omega}^\pm(g)$$

over \mathbb{C} , and let $c^\pm \in \text{GL}_n(R_0)^\Sigma$ satisfy

$$\underline{\omega}_{\underline{A}/R}^\pm = c^\pm \cdot \underline{\omega}_{can}^\pm(\underline{A})$$

over R_0 , where $\underline{\omega}_{can}^\pm$ is the basis with this notation in [Eis12, Section 5]. (For the results in the current paper, the precise definition of $\underline{\omega}_{can}^\pm$ is unnecessary.) Then by application of the main results on algebraicity in [Eis12],

$$\begin{aligned} & \det(\Omega^+)^{-(k+\nu+d)} \det(\Omega^-)^{-(\nu-d)} \det(z - {}^t\bar{z})^{-d} \prod_{\sigma \in \Sigma} (2\pi i)^{nd(\sigma)} \Psi_{n,d,k} D(1, K, \mathfrak{b}, p, k)^{-1} E_{k+2d, \nu-d} \left(z; h, \chi, \mu, \frac{k}{2} \right) \\ (36) \quad & = \det(c^+)^{-(k+\nu+d)} \det(c^-)^{-(\nu-d)} \theta^d G_{k, \nu, \chi, \mu}(\underline{A}). \end{aligned}$$

(In [EHLS], we explain how to choose CM periods Ω^\pm and c^\pm uniformly for all CM abelian varieties at once. For the current paper, however, this will not be necessary.)

Since we started with functions on groups, we conclude this subsection by reminding the reader of the simple relationship

$$E_{k+2d, \nu-d} \left(z; h, \chi, \mu, \frac{k}{2} \right) = j_{g_\infty}^{k+2d, \nu-d}(\mathfrak{z}) E_f(m(h)g_\infty),$$

where $h \in \mathrm{GL}_n(\mathbb{A}^\infty)$ and $g_\infty \in G(\mathbb{R})^\Sigma$ such that $g_\infty \mathfrak{z} = z$.

Highest weights and higher dimensional representations. The differential operators discussed above are a special case of certain differential operators that map the Eisenstein series to automorphic forms whose weight is a representation of dimension ≥ 1 . (See, for example, [Shi00] for a description of these more general operators in the C^∞ -case. The author's thesis, [Eis12], is the reference for these operators in the p -adic setting.)

This special case covered in the present paper is sufficient for p -adically interpolating certain special values of non-holomorphic Eisenstein series $E_{k, \nu}(z; h, \chi, \mu, s)$ with s not necessarily $\frac{k}{2}$ and k not necessarily an integer (but rather a Σ -tuple of integers). This is enough for both [EHLS] and for the current state of homotopy theory.

In [Eis], we use these operators to generalize the results of this paper to automorphic forms of non-scalar weights. We obtain a similar result to Equation (34) but with $\Psi_{n, d, k}$ replaced by a number in terms of the highest weight of a given representation. The results of this paper naturally generalize to the setting of [Eis], but for simplicity, we have divided them into two portions.

Lie theory. There is a Lie-theoretic approach to the C^∞ -differential operators that might be more natural in the context of automorphic forms on adèle groups. (See, for example, the appendix to [Shi00].) We have used the above approach due to the fact that there are currently no references for a p -adic analogue of the C^∞ -Lie-theoretic differential operators. The author hopes to fill in this gap in the literature in the near future.

4. A p -ADIC EISENSTEIN MEASURE

Let $T(r)$ be as in Equation (30). Let

$$\mathcal{G} = ((\mathcal{O}_K \otimes \mathbb{Z}_p)^\times / \overline{\mathcal{O}_K^\times}) \times T(r),$$

where $\overline{\mathcal{O}_K^\times}$ denotes the closure of \mathcal{O}_K^\times in $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times$. Let \mathcal{V} denote the space of p -adic automorphic forms, as above. From any Hecke character χ , we obtain a p -adic character $\tilde{\chi}$ in the notation of Equation (32) by shifting to p .

Recall that in Sections 2.2.11 through 2.2.13, we introduced Eisenstein series and $E_{F, \mu}$ whose q -expansions were shown in Equations (29)

and (33) to be

$$G_{k,\nu,\chi,\mu}(q) = \sum_{\beta} \left(\alpha(\beta, \chi) \phi_{(\chi_1^{-1}\chi_2\mu)}({}^t\beta) \prod_{v \in \Sigma} \sigma_v(\det \beta)^{k-n} \right) q^{\beta}$$

$$E_{F,\mu}(q) = \sum_{\beta} \left(\left(\prod_{v \nmid p^\infty} P_{\beta,v,b}(F(\pi_v)) \right) \cdot F(\det \beta^{-1}) N_{E/\mathbb{Q}}(\det \beta)^{-n} \phi_\mu(\beta) \right) q^{\beta}.$$

Theorem 20 (An Eisenstein measure). *There is a continuous \mathcal{V} -valued measure $d\phi$ on \mathcal{G} such that*

$$\int_{\mathcal{G}} (\tilde{\chi}_p, \mu \prod_{\sigma \in \Sigma} \sigma^{d(\sigma)}) d\phi = \theta^d G_{k,\nu,\chi',\mu} = E_{\tilde{\chi},\mu \cdot \prod_{\sigma \in \Sigma} \sigma^{d(\sigma)}}$$

for all locally constant characters μ on $T(r)$ and Hecke characters χ with infinity type $\psi_{k,\nu} = \prod_{\sigma \in \Sigma} \sigma^{-k} \left(\frac{\sigma}{\sigma} \right)^\nu$ and conductor dividing p^∞ , where $\chi' := \chi \cdot |\cdot|_{K/\mathbb{Q}}^{\frac{k}{2}}$.

Proof. From the expression of the Fourier coefficients in Example 19 and the discussion at the end of Section 2.2.12, it immediately follows that there is a \mathcal{V} -valued measure $d\phi$ on \mathcal{G} given by

$$\int_{\mathcal{G}} (F, \mu) d\phi = E_{F,\mu}$$

for each continuous $\mathcal{O}_{\mathbb{C}_p}$ -valued function F on $((\mathcal{O}_K \otimes \mathbb{Z}_p)^\times / \overline{\mathcal{O}_K^\times})$ extended by 0 to $\mathcal{O}_K \otimes \mathbb{Z}_p$ and each continuous $\mathcal{O}_{\mathbb{C}_p}$ -valued function μ on $T(r)$. The theorem now follows directly from Equation (35) and Example 19 (together with the q -expansion principle), by taking $F = \tilde{\chi}$. \blacksquare

As a direct corollary of Theorem 20, we obtain:

Corollary 21. *Let R be an \mathcal{O}_K -subalgebra of $\mathcal{O}_{\mathbb{C}_p}$. For each point \underline{A} consisting of an ordinary abelian variety with PEL structure (i.e. a point of the Igusa tower) defined over R , together with a choice of basis for $\underline{\omega} = \underline{\omega}_{\underline{A}/R}$, there is a $\mathcal{O}_{\mathbb{C}_p}$ -valued measure $d\phi(\underline{A}, \underline{\omega})$ on \mathcal{G} defined by*

$$\int_{\mathcal{G}} (F, \mu) d\phi(\underline{A}, \underline{\omega}) = E_{F,\mu}(\underline{A}, \underline{\omega})$$

for all $\mathcal{O}_{\mathbb{C}_p}$ -valued continuous functions (F, μ) on \mathcal{G} .

Technically, the above theorem and corollary were valued in the p -adic automorphic forms restricted to points of the Shimura variety for which $m(\prod_{v \nmid p^\infty} h_v) = 1_{2n} \in M(\mathbb{A}^{p,\infty})$. However, the above results naturally generalize to the ring of p -adic automorphic forms even after

we remove the restriction $m(\prod_{v \nmid p^\infty} h_v) = 1_{2n} \in M(\mathbb{A}^{p,\infty})$. Thus, we generalize Theorem 20 and Corollary 21 to obtain measures on

$$\mathcal{G}' := \mathcal{U} \times T(r),$$

where \mathcal{U} is the idèle class group $H(p^\infty) \xrightarrow{\sim}$ the maximal abelian extension of K of conductor p^∞ . We identify $((\mathcal{O}_K \otimes \mathbb{Z}_p)^\times / \overline{\mathcal{O}_K^\times})$ with its image in \mathcal{U} and note that $\mathcal{U} / ((\mathcal{O}_K \otimes \mathbb{Z}_p)^\times / \overline{\mathcal{O}_K^\times})$ is finite.

Corollary 22 (Improvement of Theorem 20). *There is a continuous \mathcal{V} -valued measure $d\phi$ on \mathcal{G}' such that the q -expansion of*

$$\int_{\mathcal{G}'} (\tilde{\chi}, \mu \prod_{\sigma \in \Sigma} \sigma^{d(\sigma)}) d\phi$$

is $\theta^d G_{k,\nu,\chi',\mu}(\bullet, q) = E_{\tilde{\chi}, \mu \cdot \prod_{\sigma \in \Sigma} \sigma^{d(\sigma)}}(\bullet, q)$ for all locally constant characters μ on $T(r)$ and Hecke characters χ with infinity type $\psi_{k,\nu} = \prod_{\sigma \in \Sigma} \sigma^{-k} \left(\frac{\bar{\sigma}}{\sigma}\right)^\nu$ and conductor dividing p^∞ , where $\chi' := \chi|_{\cdot|_{K/\mathbb{Q}}}^{\frac{k}{2}}$. The first variable \bullet denotes a point of $M(\mathbb{A}^{p,\infty})$.

Proof. Apply Lemma 9 and Theorem 20. ■

Theorem 23 (p -adic interpolation of special values of C^∞ -Eisenstein series). *Let $d = (d(\sigma))_{\sigma \in \Sigma} \in \mathbb{Z}^\Sigma$, and let R be an \mathcal{O}_K subalgebra of \mathbb{C} such that there is also an injection $R \hookrightarrow R_0 = \varprojlim_n R/p^n R$. For each CM point $\underline{A}(g)$ defined over R corresponding to a point $g \in \text{Sh}(\mathbb{C})$ and satisfying conditions (†) and (‡) as in Section 3.0.14,*

$$\frac{\int_{\mathcal{G}'} (\tilde{\chi}, \mu \cdot \prod_{\sigma \in \Sigma} \sigma^{d(\sigma)}) d\phi(\underline{A}(g))}{\det(c^+)^{k+\nu+d} \det(c^-)^{-(\nu-d)}} = \frac{(2\pi i)^{nd} \Psi_{n,d,k} \det(z - {}^t \bar{z})^{-d}}{\det(\Omega^+)^{k+\nu+d} \det(\Omega^-)^{-(\nu-d)} D(n, K, \mathfrak{b}, p, k)} E_{k+2d, \nu-d} \left(z; \prod_{v \nmid p^\infty} g_v, \chi', \mu, \frac{k}{2} \right)$$

for all locally constant characters μ on $T(r)$ and Hecke characters χ with infinity type $\psi_{k,\nu} = \prod_{\sigma \in \Sigma} \sigma^{-k} \left(\frac{\bar{\sigma}}{\sigma}\right)^\nu$ and conductor dividing p^∞ , where $\chi' := \chi|_{\cdot|_{K/\mathbb{Q}}}^{\frac{k}{2}}$ and $z = g_\infty \mathbf{1}$.

Proof. This follows from Theorem 20 and its corollaries, together with Equation (36). ■

REFERENCES

- [AHR10] Matthew Ando, Michael Hopkins, and Charles Rezk, *Multiplicative orientations of KO-theory and of the spectrum of topological modular forms*, 2010, <http://www.math.uiuc.edu/~mando/papers/koandtmf.pdf>.
- [Beh09] Mark Behrens, *Eisenstein orientation*, 2009, Typed notes available at <http://www-math.mit.edu/~mbehrens/other/coredump.pdf>.

- [EHLS] Ellen E. Eischen, Michael Harris, Jian-Shu Li, and Christopher M. Skinner, *p-adic L-functions for unitary groups*, In preparation.
- [Eis] Ellen E. Eischen, *A higher-dimensional p-adic Eisenstein measure for unitary groups*, In preparation.
- [Eis12] ———, *p-adic differential operators on automorphic forms on unitary groups*, *Annales de l'Institut Fourier* **62** (2012), To appear, also available at <http://arxiv.org/abs/1006.4898>, 68 pages.
- [Har81] Michael Harris, *Special values of zeta functions attached to Siegel modular forms*, *Ann. Sci. École Norm. Sup. (4)* **14** (1981), no. 1, 77–120. MR MR618732 (82m:10046)
- [Har86] ———, *Arithmetic vector bundles and automorphic forms on Shimura varieties. II*, *Compositio Math.* **60** (1986), no. 3, 323–378. MR MR869106 (88e:11047)
- [HLS06] Michael Harris, Jian-Shu Li, and Christopher M. Skinner, *p-adic L-functions for unitary Shimura varieties. I. Construction of the Eisenstein measure*, *Doc. Math.* (2006), no. Extra Vol., 393–464 (electronic). MR MR2290594 (2008d:11042)
- [Hop95] Michael J. Hopkins, *Topological modular forms, the Witten genus, and the theorem of the cube*, *Proceedings of the International Congress of Mathematicians*, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 554–565. MR 1403956 (97i:11043)
- [Hop02] M. J. Hopkins, *Algebraic topology and modular forms*, *Proceedings of the International Congress of Mathematicians*, Vol. I (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 291–317. MR 1989190 (2004g:11032)
- [Kat78] Nicholas M. Katz, *p-adic L-functions for CM fields*, *Invent. Math.* **49** (1978), no. 3, 199–297. MR MR513095 (80h:10039)
- [Lan08] Kai-Wen Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, 2008, Ph.D. thesis, Harvard University, available at <http://www.math.princeton.edu/~klan/articles/cpt-PEL-type-thesis-single.pdf>.
- [Lan11] ———, *Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties*, *Crelle's Journal* (2011), To appear.
- [Shi82] Goro Shimura, *Confluent hypergeometric functions on tube domains*, *Math. Ann.* **260** (1982), no. 3, 269–302. MR 669297 (84f:32040)
- [Shi83] ———, *On Eisenstein series*, *Duke Math. J.* **50** (1983), no. 2, 417–476. MR 705034 (84k:10019)
- [Shi84] ———, *On differential operators attached to certain representations of classical groups*, *Invent. Math.* **77** (1984), no. 3, 463–488. MR 759261 (86c:11034)
- [Shi97] ———, *Euler products and Eisenstein series*, *CBMS Regional Conference Series in Mathematics*, vol. 93, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1997. MR MR1450866 (98h:11057)
- [Shi00] ———, *Arithmeticity in the theory of automorphic forms*, *Mathematical Surveys and Monographs*, vol. 82, American Mathematical Society, Providence, RI, 2000. MR MR1780262 (2001k:11086)
- [Tan99] Victor Tan, *Poles of Siegel Eisenstein series on $U(n, n)$* , *Canad. J. Math.* **51** (1999), no. 1, 164–175. MR 1692899 (2000e:11073)